Introduction to algorithmic mechanism design

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Part I

Game Theory and Computer Science

Why Game Theory and Computer Science?

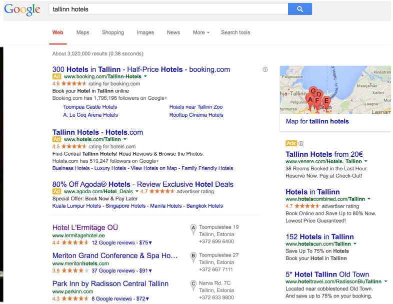
Past

Present

Algorithmic game theory

Some applications

- Internet routing (interactions between ISPs)
- Sponsored search
- Online auctions (e.g. Ebay)
- P2P (e.g. free-riders)
- ...



Algorithmic Game Theory

Main research directions

- Computational issues of games (e.g., finding any Nash equilibrium is PPAD-complete)
- Price of Anarchy (inefficiency of systems with selfish entities)
- Algorithms + Incentives = Mechanisms

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Notions of equilibria

Nash equilibrium

No player has reasons to deviate.

- Advantages: Natural. It always exists.
- Disadvantages: More than one. Computationally hard?

Dominant strategies

Each player has an optimal strategy, no matter what.

- Advantages: Natural. Great for implementation.
- Disadvantages: Rarely exists.

- In these lectures we consider dominant equilibria.

Mechanism design

Mechanisms as algorithms

- Given an objective, design **a game** whose equilibrium optimizes the objective.

Objectives

Usually we want to optimize one of the following:

- Revenue (sum of payments)
- Social welfare (sum of player values)
- Other (for example, makespan)

Typical example: single-item auction

Problem

- *We want to sell an object to n bidders (buyers).*
- *Each bidder has a value v_i for the object, which is known only to him/her.*
- **Objective:** *Social welfare, equivalent to "give the item to the bidder with the highest value".*

Features

- Incomplete information: only the bidders know their values
- Money may be used as an incentive. But, money may not be part of the objective.
- Direct revelation: The bidders declare all their values at the beginning.

Example: Single-item auction (cont.)

Auctions for maximizing welfare:

- Each bidder declares a value $\tilde{v}_i$, not necessarily equal to the true value v_i.
- The mechanism allocates the object to the bidder with the highest bid, $\max_i \tilde{v}_i$. *This is the objective when the bidders are truthful.*
- **First-price auction:** The bidder pays **her bid**.
- **Vickrey auction:** The bidder pays only the **second highest bid**.

Proposition: The Vickrey auction mechanism is truthful, but the first-price auction is not.

Why is the Vickrey auction truthful?

- The payment depends only on the other players
- The allocation is monotone: increasing the declared value makes it more likely to get the item

Part II

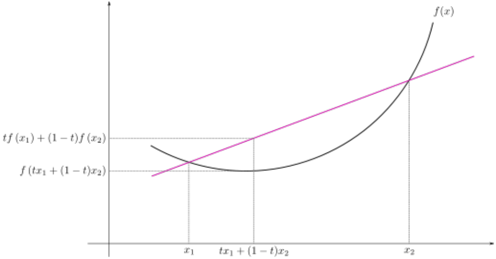
Convexity

Convexity

Definition: A function $f : R^n \to R$ is called convex when

$$\lambda f(x) + (1-\lambda)f(y) \geq f(\lambda x + (1-\lambda)y)$$

for every x, y and $\lambda \in [0, 1]$.



The three layers of convexity

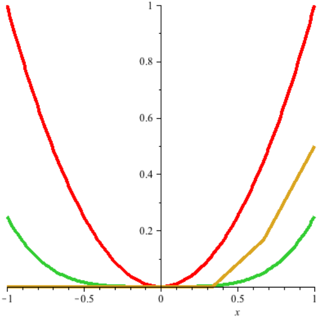
We focus on functions of `one variable`, but everything generalizes appropriately to many variables.

The following are equivalent (for doubly differentiable functions)

1. $f(x)$ is convex
2. $f'(x)$ is monotone (nondecreasing)
3. $f''(x)$ is nonnegative

Examples of convex functions

- x^2
- $\frac{1}{4}x^4$
- $\max\{0, \frac{x}{2} - \frac{1}{6}, x - \frac{1}{2}\}$



Important properties of convex functions

Proposition

For every function g, the function f define by

$$f(x) = \sup_{y}\{x \cdot y - g(y)\},$$

is convex.

Proposition

For every convex function f, there exists a function f^* (called the conjugate of f), such that

$$f(x) = \sup_{y}\{f'(y) \cdot x - f^*(f'(y))\}$$

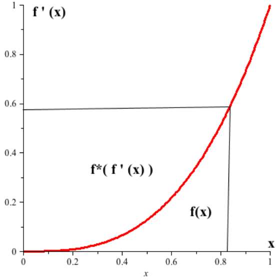
Both propositions hold for functions of many variables. Simply interpret "·" as inner product, and f' as ∇f. For example:

$$f(x) = \sup_{y}\{\nabla f(y) \cdot x - f^*(\nabla f(y))\}$$

Conjugate

The conjugate function f^* of a function f is defined by

$$f^*(y) = \sup_x \{x \cdot y - f(x)\}$$



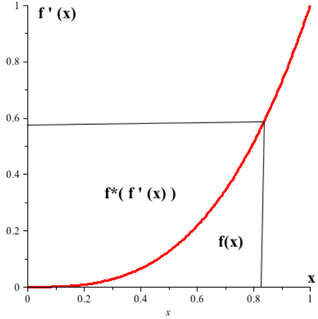
Notice the symmetry

$$f^*(y) = \sup_x \{x \cdot y - f(x)\} \qquad x \leftrightarrow y \quad \text{where } y = f'(x)$$

$$f(x) = \sup_y \{x \cdot y - f^*(x)\} \qquad f \leftrightarrow f^*$$

Example

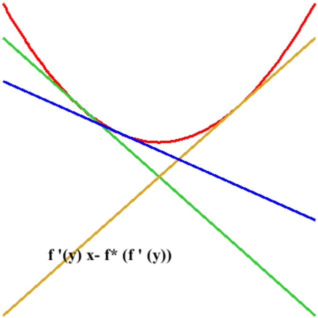
- $f(x) = \frac{1}{4}x^4$
- $f'(x) = x^3$
- $f^{*\prime}(x) = x^{1/3}$
- $f^*(x) = \frac{3}{4}x^{4/3}$
- $f'(x) \cdot x - f^*(f'(x)) = x^3 \cdot x - \frac{3}{4}(x^3)^{4/3} = f(x)$



Supporting hyperplanes

For every convex function f, the conjugate function f^* defines the supporting hyperplanes

$$f(x) = \sup_y \{ f'(y) \cdot x - f^*(f'(y)) \}$$



Example:

$$f(x) = \frac{1}{4}x^4 = \sup_y \left\{ y^3 \cdot x - \frac{3}{4}y^4 \right\}.$$

Part III

Truthfulness, monotonicity, and convexity

Truthfulness in mechanism design

- A mechanism is truthful when the participants have no incentive to lie
- Same notions: truthful, incentive compatible, strategyproof

Single-bidder single-item auction

- We have an item and a bidder (potential buyer)
- The bidder has a private value v for the item
- We want to design an "auction" in which the bidder reveals her value v
- The mechanism consists of 2 functions:
  - An allocation function a and a payment function p
- Protocol:
  - We announce the mechanism (a, p)
  - The bidder declares her value $\tilde{v}$ (perhaps a lie)
  - We give the item to the bidder with probability $a(\tilde{v})$ and she pays $p(\tilde{v})$.

Truthful?

- The utility of the player is

$$u(\tilde{v}|v) = a(\tilde{v}) \cdot v - p(\tilde{v})$$

- For which functions a and p is the mechanism truthful?
- A mechanism is truthful when $u(v|v) = \sup_{\tilde{v}} u(\tilde{v}|v)$.
- Two types of truthfulness:

Truthful	Truthful in expectation
$a(v) \in \{0, 1\}$	$a(v) \in [0, 1]$
deterministic	randomized

Truthful in expectation is not ex post truthful.

Truthful = convex utilities

Theorem

A mechanism is truthful if and only if the utility $u(v) = u(v|v)$ of the bidder is a convex function of her private value v.
Furthermore, the probability of getting item j is given by

$$a(v) = u'(v)$$

The payment is given by $p(v) = u^*(u'(v))$.

For auctions of many items, in which the bidder has a vector v of private values (one for each item) the same holds. Specifically,

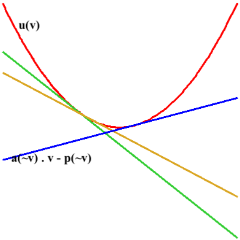
$$a(v) = \nabla u(v)$$
$$p(v) = u^*(\nabla u(v))$$

Truthfulness implies convexity:

The utility of every mechanism is a convex function!

$$u(v) = \sup_{\tilde{v}}\{u(\tilde{v}|v)\} = \sup_{\tilde{v}}\{a(\tilde{v}) \cdot v - p(\tilde{v})\},$$

a convex function of v. This holds even for non-truthful mechanisms.



Furthermore, since $u(v) = \sup_y\{u'(y) \cdot v - u^*(u'(y))\}$:

- Allocation probability: $a(v) = u'(v)$
- Payment: $p(v) = u^*(u'(v))$

Convexity implies truthfulness:

Assume that

- $u(v)$ convex
- $u'(v)$ satisfy the allocation constraints: $0 \leq u'(v) \leq 1$

Then convexity implies

$$\begin{aligned} u(v|v) &= u(v) \\ &= \sup_{\tilde{v}} \{u'(\tilde{v}) \cdot v - u^*(u'(\tilde{v}))\} \\ &= \sup_{\tilde{v}} \{u(\tilde{v}|v)\} \end{aligned}$$

which implies truthfulness.

Main points on truthfulness

- In many computational problems, the input is controlled by selfish entities
- The algorithm must provide incentives to these entities to be truthful
- A necessary and sufficient condition is that the algorithm must be monotone
- This entails two conditions:
  - The solution must be the gradient of the utilities of the entities
  - The utilities must be convex
- The payments can be computed directly from the monotone solution