Introduction to algorithmic mechanism design

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Profit maximization in auctions

$$v_1 = 12$$
$$v_2 = 7$$
$$v_3 = 6$$

Objective: Maximize revenue

Vickrey auction

$$v_1 = 12$$
$$v_2 = 7$$
$$v_3 = 6$$

Vickrey auction: Give the item to the highest bidder (bidder 1). The winner pays the second highest bid (payment=7).

A better auction for maximizing revenue?

- The revenue of the Vickrey auction may be far from optimal
  - For example, when $v_1 = 1000$, $v_2 = 10$
- On the other hand, first-price auction can be strategically manipulated
  - In the example, the first player will bid $\tilde{v}_1 = 11$ (assuming complete information for the bidders)

Optimizing with incomplete information

- Because the auctioneer has incomplete information, it is not even clear how to formulate the optimization problem of maximizing revenue
- There are two approaches to address this issue
- Worst-case approach
  - We make as few assumptions as necessary and compute the optimal solutions assuming an adversarial setting
  - Advantage: Few assumptions
  - Disadvantage: Pessimistic
- Bayesian approach
  - We assume that the unknown parameters are drawn from publicly-known probability distributions
  - Advantage: Reasonable and "practical"
  - Disadvantage: Based on arbitrary strong assumptions

The Bayesian setting for single item auctions

- The value of bidder i comes from a probability distribution F_i
- All the distributions are known by the bidders and the auction designer
- For example, there are n bidders with values independently drawn from the uniform distribution $U[0, 1]$

One bidder, single item

- Assume that we have a single bidder with value $v \sim F$
- Deterministic auctions: Post a price p
  - If $v \geq p$, the bidder gets the item and pays p
  - Otherwise, the bidder does not get the item and pays 0
- What is the optimal p?
  - p maximizes $p(1 - F(p))$
  - For example, for the uniform distribution $U[0, 1]$: $p = 1/2$ and the expected revenue is $1/4$

First price auction?

- Assume n bidders with values drawn from distributions F_i
- What is a **Bayesian Nash equilibrium** for the first-price auction?
- Not an easy answer
- For example, with 2 bidders with v_1, $v_2 \sim U[0, 1]$:
  - Each bidder lies and declares half of her actual value: $b_i(v_i) = v_i/2$
  - Even with lies, the highest bid gets the item!

Revenue equivalence principle

- Assume 2 bidders with values drawn from distribution $U[0,1]$
- With the first-price auction the expected revenue is $1/3$
  - equal to $E_{v_1,v_2 \sim U[0,1]}[\max(v_1,v_2)/2]$, because bidders declare half of their actual values
- With the second-price (Vickrey) auction the expected revenue is also $1/3$
  - equal to $E_{v_1,v_2 \sim U[0,1]}[\min(v_1,v_2)]$

Proposition

All single-item auctions that allocate the item to the player with highest value have the same expected revenue.

(Another version of: the payments are determined by the allocation)

Better than Vickrey?

- Assume bidders with values drawn from distribution $U[0,1]$
- Is there an auction with higher revenue than the Vickrey auction?
- Yes: Vickrey auction with **reserve price** $1/2$.
  - It gives the item to the highest bidder only if her value exceeds the reserve price
  - She pays the maximum of the second bid and $1/2$
  - Equivalent to: the auctioneer participates in the auction as a bidder with value equal to the reserve price
- Revenue: $5/12$ (better than $1/3$)

Problem: Given the probability distributions of the values, find the auction which maximizes revenue.

Optimal auctions for single item

- There are n bidders with values v_i drawn from distributions F_i
- We want to design a truthful auction which maximizes revenue
- The revelation principle holds: every auction can be transformed into an equivalent truthful one

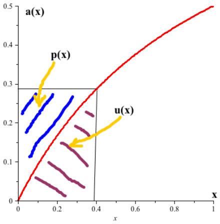
Optimal auctions for single item

- v_i : value of bidder i
- F_i : cumulative probability distribution of v_i, $F_i(x) = \mathbf{Pr}(v_i \leq x)$
- f_i : probability density function $f_i(x) = F_i'(x)$
- $a_i(v)$: allocation probability, i.e. probability of bidder i getting the item
- $p_i(v)$: payment of bidder i

Theorem

An auction is truthful if and only if the allocation probability $a_i(v)$ is non-decreasing in v_i. The payment for player i is given by

$$p_i(v) = a_i(v) \cdot v_i - \int_0^{v_i} a_i(z)\, dz$$



Virtual valuations

Definition

The virtual valuation of bidder i is

$$\varphi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

Theorem (Myerson)

The expected profit of a truthful mechanism is equal to

$$E_v\left[\sum_i \varphi_i(v_i) a_i(v)\right]$$

Therefore, the optimal auction is **VCG applied to virtual values**!
Caution! Virtual values can be negative.

Myerson's optimal auction

- The bidders provide their bids v_i
- The auctioneer computes the virtual valuations $\varphi_i(v_i)$
- She runs VCG on the virtual valuations (discarding negative valuations)
  - She gives the item to the bidder with highest virtual valuation (if it is non-negative)
  - Let p'_i be the maximum of the second highest valuation and 0
  - The winner pays $p_i = \varphi_i^{-1}(p'_i)$

Example

- n bidders with values in $U[0,1]$
- The virtual values are

$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} = v_i - \frac{1 - v_i}{1} = 2v_i - 1$$

- Notice that $\varphi_i(1/2) = 0$; $1/2$ is a reserve price!
- Myerson's auction gives the item to the highest bidder i only if $v_i \geq 1/2$
- The winner pays the maximum of the second highest bid and the reserve price

Ironing for irregular distributions

- The auction computes payments $p_i = \varphi_i^{-1}(p'_i)$
- This works only if the inverse function φ_i^{-1} is defined, or equivalently if φ_i is a monotone function.
- Otherwise, we have to do some **"ironing"** of the functions

Proof of Myreson's theorem

The payment of bidder i when her value is v_i is given by

$$p_i(v) = a_i(v) \cdot v_i - \int_0^{v_i} a_i(z)\,dz$$

For simplicity, assume that the values are in $[0, h]$. The expected revenue from bidder i

$$\begin{aligned} R_i &= \int_0^h p_i(v) f_i(v_i)\,dv_i \\ &= \int_0^h a_i(v) \cdot v_i \cdot f_i(v_i)\,dv_i - \int_0^h \int_0^{v_i} a_i(z) f_i(v_i)\,dz\,dv_i \end{aligned}$$

The whole trick is to inverse the order of the two integrals in the above expression. By doing this we get

$$R_i = \int_0^h \left(v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right) a_i(v_i) f_i(v_i)\,dv_i = E_{v_i}[\varphi_i(v_i) a_i(v)]$$

Adding the revenue from all bidders, we get the theorem.

Two or more items

- Auctions with 2 or more items are notoriously hard
- We don't even know the optimal auction for the special case of
  - uniform distribution, additive valuations
  - 1 bidder !
  - 3 or more items (recently, with my student Yiannis Giannakopoulos, we determined the optimal auction for up to 6 items)