XX Eesti Arvutiteaduse Talvekool, 2015

Cellular automata, tilings and (un)computability, homework

1. Let $G : \{0,1\}^{\mathbb{Z}} \longrightarrow \{0,1\}^{\mathbb{Z}}$ be the elementary CA whose local rule is the majority rule: $f(L,C,R) = 1$ if and only if $L + C + R \geq 2$.
   (a) What is the Wolfram number of this CA ?
   (b) Prove that if the state of a cell remains unchanged in two consecutive generations then it remains unchanged from that moment on.
   (c) Prove that under iterations of G^2 the state of each cell changes at most once.
   (d) Determine all fixed points of G,
   (e) Determine which configurations c have a converging orbit $c, G(c), G^2(c), \ldots$.
2. Prove that every fully periodic (in space) configuration is eventually periodic (in time).
3. Let G be a one-dimensional CA. Prove that if a spatially periodic configuration has a pre-image then it has a spatially periodic pre-image.
4. Let us call a one-dimensional configuration $c \in S^{\mathbb{Z}}$ *rich* if it contains a copy of every finite pattern over state set S. Let $G : S^{\mathbb{Z}} \longrightarrow S^{\mathbb{Z}}$ be a surjective CA function. Prove that c is rich if and only if $G(c)$ is rich.
5. Let $A : \{0,1\}^{\mathbb{Z}^2} \longrightarrow \{0,1\}^{\mathbb{Z}^2}$ be the 2-dimensional, two-state CA with the Moore neighborhood whose local rule f is the majority rule:

$$f(a,b,c,d,e,f,g,h,i) = \begin{cases} 0, & \text{if } a+b+c+d+e+f+g+h+i \leq 4, \\ 1, & \text{if } a+b+c+d+e+f+g+h+i \geq 5. \end{cases}$$

   Find an *orphan* for this CA, that is, a finite pattern without a pre-image.
6. Consider *Game-of-Life*.
   (a) Calculate how much imbalance there is in the local rule. More precisely, count the cardinalities of $f^{-1}(0) \subseteq \{0,1\}^9$ and $f^{-1}(1) \subseteq \{0,1\}^9$, where $f : \{0,1\}^9 \longrightarrow \{0,1\}$ is the local rule of Game-of-Life.
   (b) Using the result of (a) and the technique in the proof of "imbalance $\Longrightarrow$ orphan", show that Game-of-Life has an orphan of size 40×40.
7. Toom's CA is the two-dimensional CA over the state set $\{0,1\}$, in which each cell takes the majority among its own state and the states in its upper and right neighbors. Prove that every 0-finite configuration evolves into the 0-uniform configuration, and that every 1-finite configuration evolves into the 1-uniform configuration.

8. Consider the following one-dimensional CA (due to Gacs, Kurdiumov, Levin): The state set is $S = \{\leftarrow, \rightarrow\}$. A cell changes the direction of its arrow if and only if there are opposing arrows at the first and the third neighbor on the side pointed by the arrow. In other words, the circled arrow is swapped in the following contexts:

$\rightarrow$	$\leftarrow$		$\leftarrow$

$\rightarrow$		$\rightarrow$	$\leftarrow$

Prove that every $\rightarrow$-finite configuration becomes eventually $\rightarrow$-uniform, and prove that every $\leftarrow$-finite configuration becomes eventually $\leftarrow$-uniform. (As in the two-dimensional Toom's rule in the previous problem.)

(Hint: Let c be $\rightarrow$-finite. Consider the finite patterns $p_1 = \leftarrow\ \leftarrow$ and $p_2 = \leftarrow\ \rightarrow\ \rightarrow\ \leftarrow$. Prove that the position of the leftmost symbol of the leftmost occurrence of pattern p_1 or p_2 in c moves at least one cell to the right each time step, until there is no occurrence of p_1 and p_2. Then show that once there is no p_1 and p_2, the position of the rightmost symbol $\leftarrow$ moves at least three positions to the left, until all cells are in state $\rightarrow$.)