Lecture 2: More on cellular automata

- Surjective CA: balance, Garden-of-Eden -theorem
- Injectivity and surjectivity on periodic configurations

Garden-Of-Eden and orphans

Configurations that do not have a pre-image are called **Garden-Of-Eden** -configurations. Only non-surjective CA have GOE configurations.

A finite pattern consists of a finite domain $D \subseteq \mathbb{Z}^d$ and an assignment

$$p : D \longrightarrow S$$

of states.

Finite pattern is called an **orphan** for CA G if every configuration containing the pattern is a GOE.

From the compactness of $S^{\mathbb{Z}^d}$ we directly get:

Proposition. Every GOE configuration contains an orphan pattern.

Non-surjectivity is hence equivalent to the existence of orphans.

Balance in surjective CA

All surjective CA have **balanced** local rules: for every $a \in S$

$$\left|f^{-1}(a)\right| = |S|^{n-1}.$$

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Indeed, consider a non-balanced local rule such as rule 110 where five contexts give new state 1 while only three contexts give state 0:

$$\begin{array}{ccc}
111 & \longrightarrow & 0 \\
110 & \longrightarrow & 1 \\
101 & \longrightarrow & 1 \\
100 & \longrightarrow & 0 \\
011 & \longrightarrow & 1 \\
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001 & \longrightarrow & 1 \\
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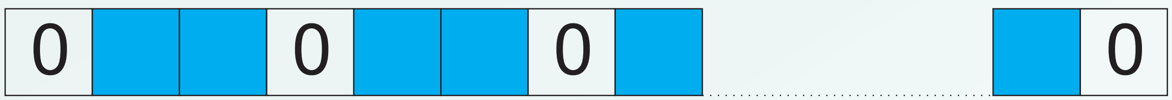
\end{array}$$

Consider finite patterns where state 0 appears in every third position. There are $2^{2(k-1)} = 4^{k-1}$ such patterns where k is the number of 0's.



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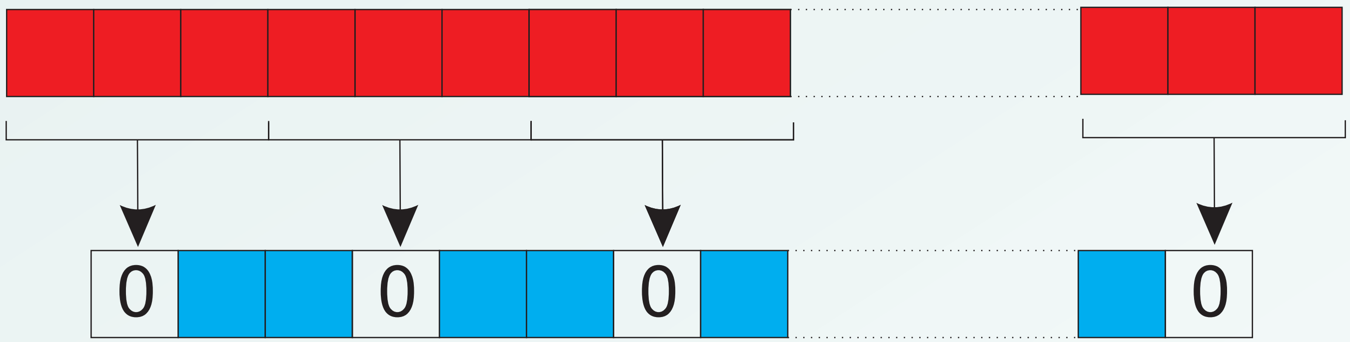


A pre-image of such a pattern must consist of k segments of length three, each of which is mapped to 0 by the local rule. There are 3^k choices.

As for large values of k we have $3^k < 4^{k-1}$, there are fewer choices for the red cells than for the blue ones. Hence some pattern has no pre-image and it must be an orphan. $\square$

One can also verify directly that pattern

$$01010$$

is an orphan of rule 110. It is the shortest orphan.

Balance of the local rule is not sufficient for surjectivity. For example, the **majority** CA (Wolfram number 232) is a counter example. The local rule

$$f(a, b, c) = 1 \text{ if and only if } a + b + c \geq 2$$

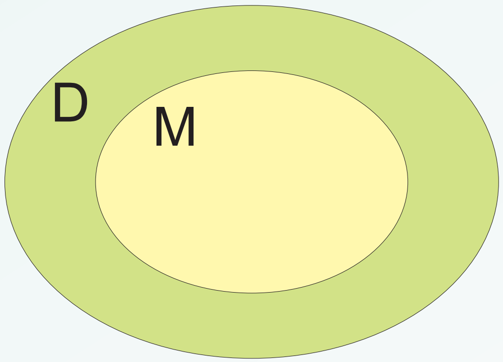
is clearly balanced, but 01001 is an orphan.

The balance property of surjective CA generalizes to finite patterns of arbitrary shape:

Theorem: Let G be surjective. Let $M, D \subseteq \mathbb{Z}^d$ be finite domains such that D contains the neighborhood of M. Then every finite pattern with domain M has the same number

$$n^{|D|-|M|}$$

of pre-images in domain D, where n is the number of states. $\square$



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The balance property means that the uniform probability measure is **invariant** for surjective CA. (Uniform randomness is preserved by surjective CA.)

Garden-Of-Eden -theorem

Let us call configurations c_1 and c_2 **asymptotic** if the set

$$diff\,(c_1, c_2) = \{\vec{n} \in \mathbb{Z}^d \mid c_1(\vec{n}) \neq c_2(\vec{n})\ \}$$

of positions where c_1 and c_2 differ is finite.

A CA is called **pre-injective** if any asymptotic $c_1 \neq c_2$ satisfy $G(c_1) \neq G(c_2)$.

The **Garden-Of-Eden -theorem** by Moore (1962) and Myhill (1963) connects surjectivity with pre-injectivity.

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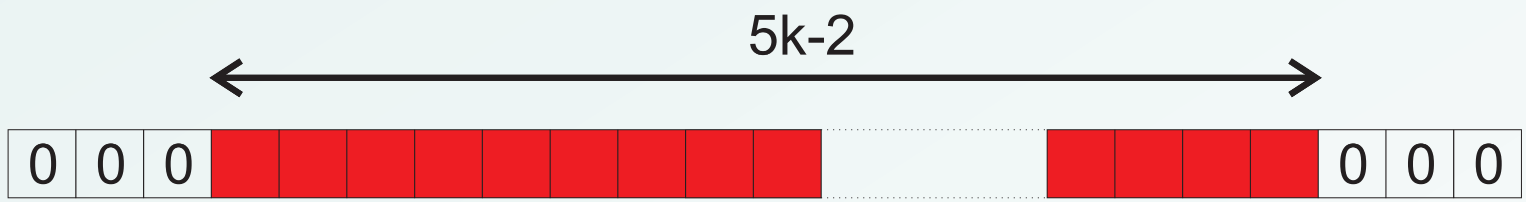
The proof idea can be easily explained using rule 110 as a running example.

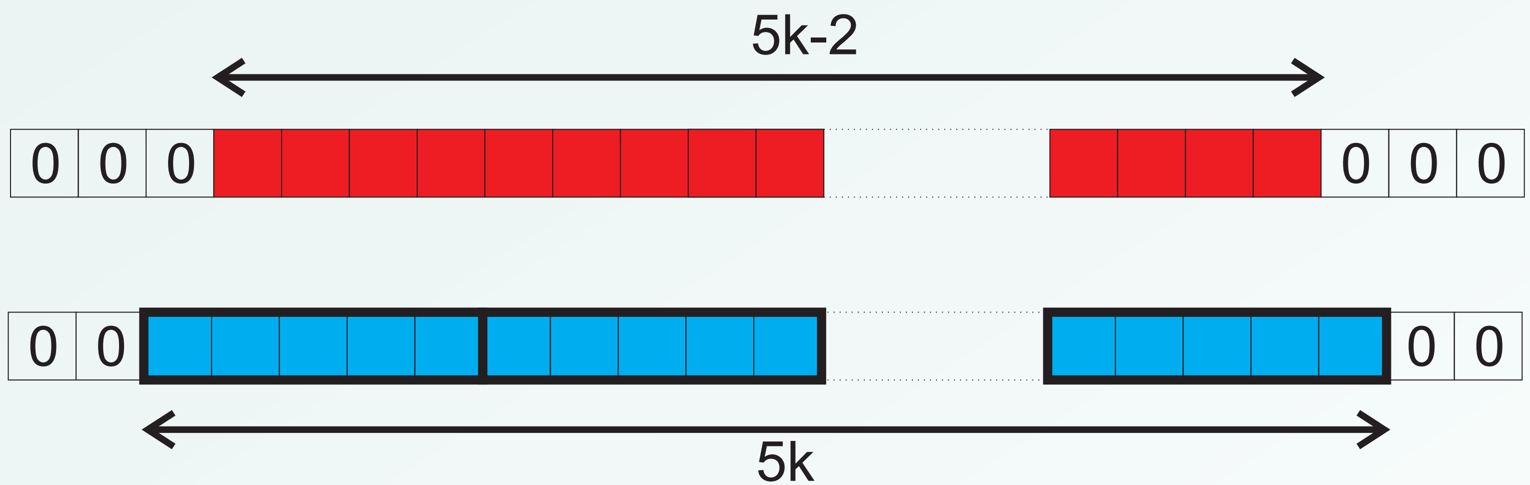
1) G not surjective $\Longrightarrow$ G not pre-injective:

Since rule 110 is not surjective it has an orphan 01010 of length five. Consider a segment of length $5k - 2$, for some k, and configurations c that are in state 0 outside this segment. There are $2^{5k-2} = 32^k/4$ such configurations.



1) G not surjective $\Longrightarrow$ G not pre-injective:

The non-0 part of $G(c)$ is within a segment of length $5k$. Partition this segment into k parts of length 5. Pattern 01010 cannot appear in any part, so only $2^5 - 1 = 31$ different patterns show up in the subsegments. There are at most 31^k possible configurations $G(c)$.



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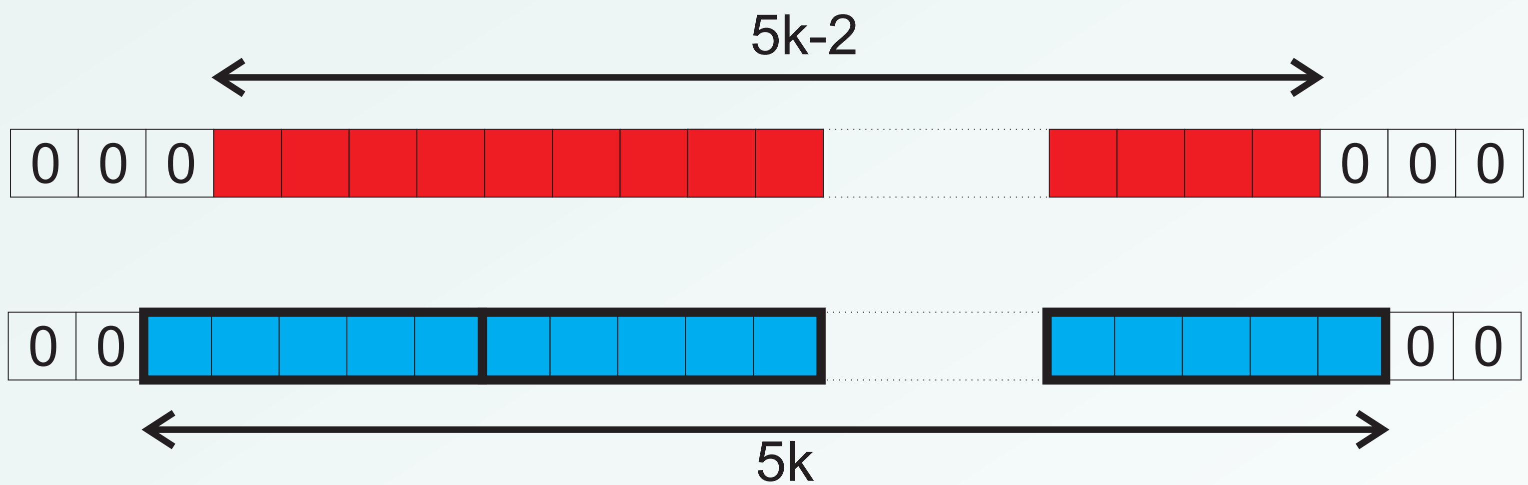
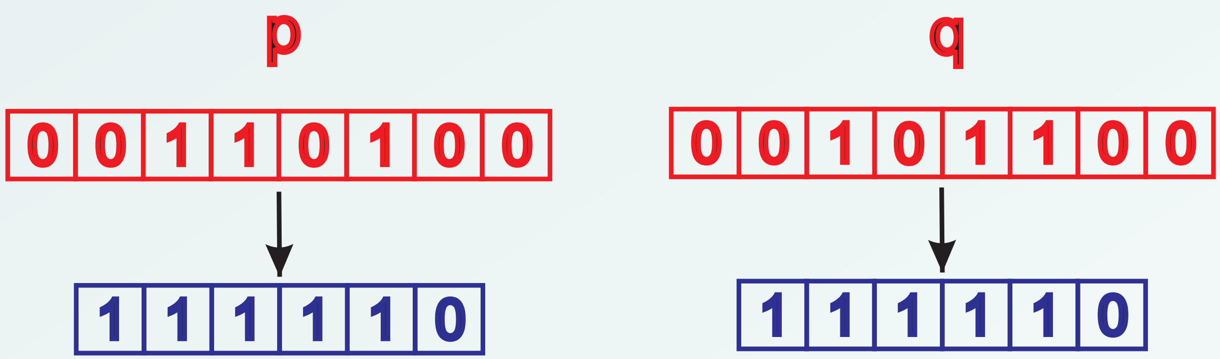


As $32^k/4 > 31^k$ for large k, there are more choices for red than blue segments. So there must exist two different red configurations with the same image. $\square$

2) G not pre-injective $\Longrightarrow$ G not surjective:

In rule 110



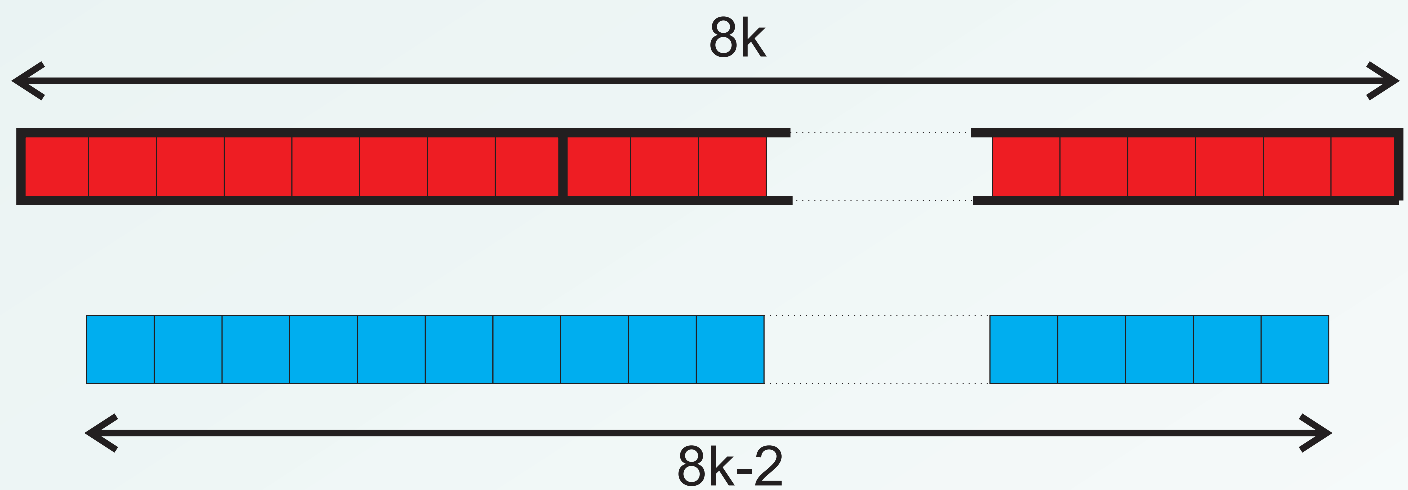
so patterns p and q of length 8 can be exchanged to each other in any configuration without affecting its image. There exist just

$$2^8 - 1 = 255$$

essentially different blocks of length 8.

2) G not pre-injective $\Longrightarrow$ G not surjective:

Consider a segment of $8k$ cells, consisting of k parts of length 8. Patterns p and q are exchangeable, so the segment has at most 255^k different images.



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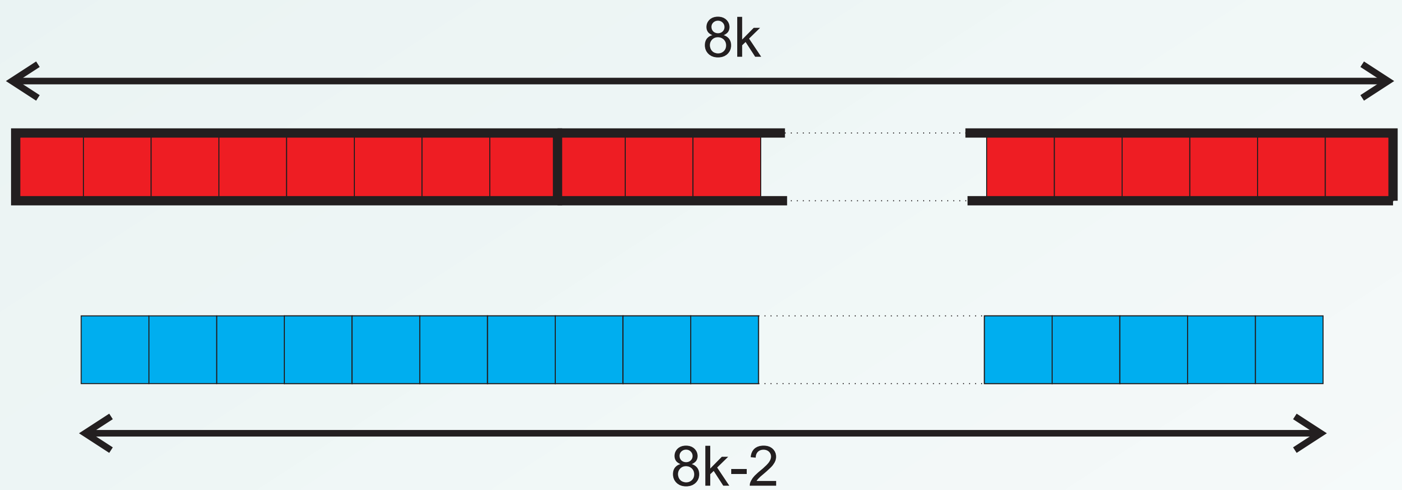

There are, however, $2^{8k-2} = 256^k/4$ different patterns of size $8k-2$. Because $255^k < 256^k/4$ for large k, there are blue patterns without any pre-image. $\square$

Garden-Of-Eden -theorem: CA G is surjective if and only if it is pre-injective.

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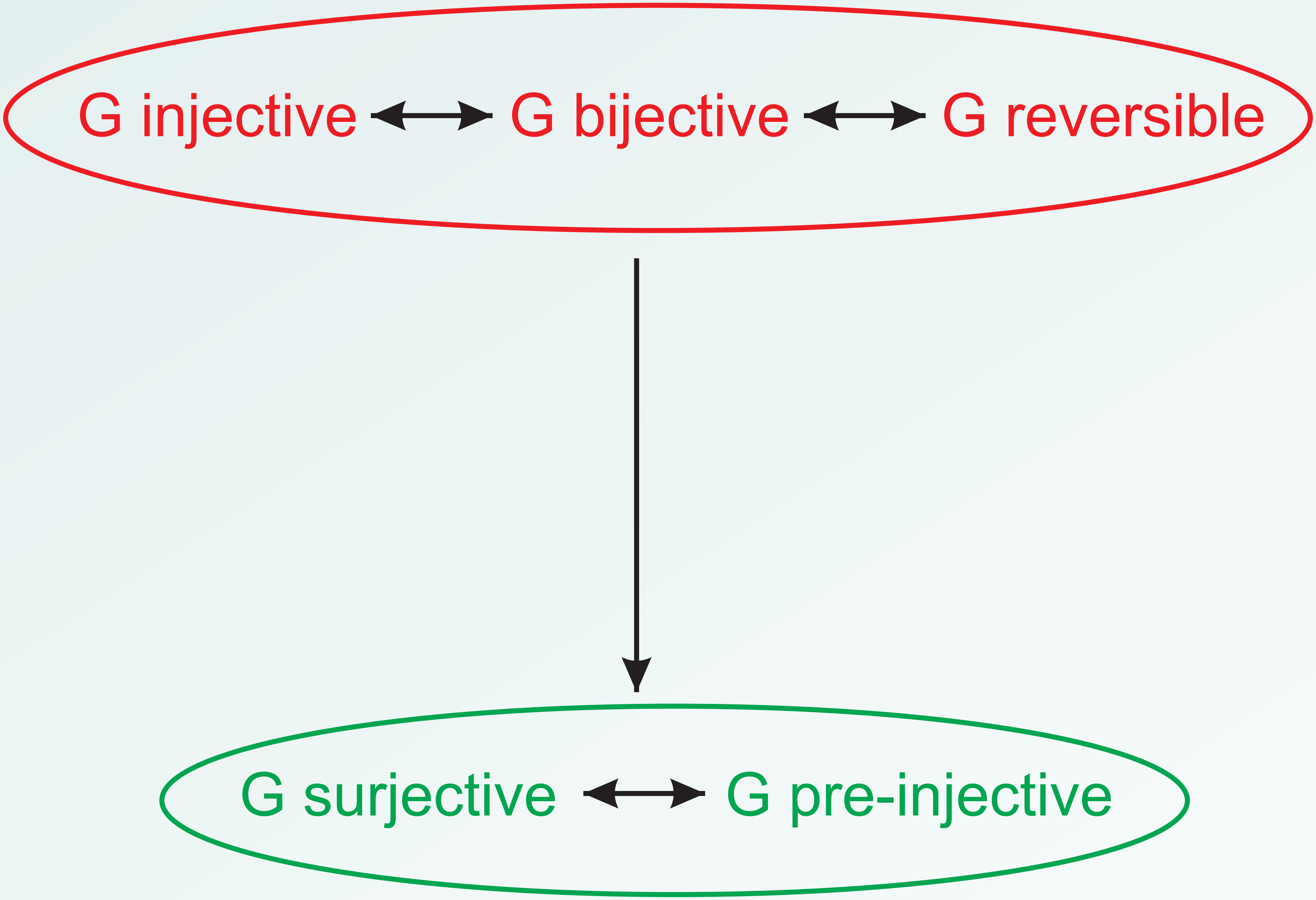
Corollary: Every injective CA is also surjective. Injectivity, bijectivity and reversibility are equivalent concepts.

Proof: If G is injective then it is pre-injective. The claim follows from the Garden-Of-Eden -theorem. $\square$


G injective
G bijective
G reversible
G surjective
G pre-injective

Examples:

The majority rule is not surjective: finite configurations

$$\ldots 0000000 \ldots \quad \text{and} \quad \ldots 0001000 \ldots$$

have the same image, so G is not pre-injective. Pattern

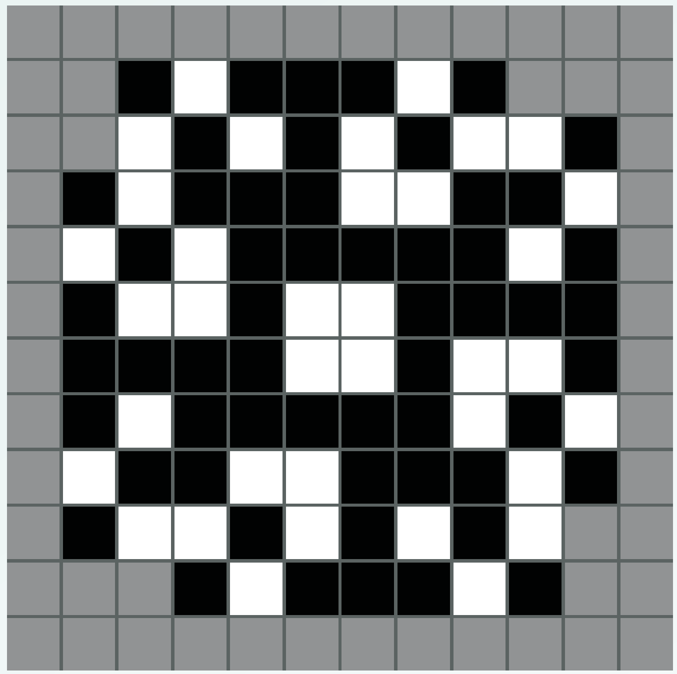
$$01001$$

is an orphan.

Examples:

In Game-Of-Life a lonely living cell dies immediately, so G is not pre-injective. GOL is hence not surjective.

Interestingly, no small orphans are known for Game-Of-Life. Currently, the smallest known orphan consists of 92 cells (56 life, 36 dead):



M. Heule, C. Hartman, K. Kwekkeboom, A. Noels (2011)

Examples:

The **Traffic CA** is the elementary CA number 226.

$$\begin{array}{ccc} 111 & \longrightarrow & 1 \\ 110 & \longrightarrow & 1 \\ 101 & \longrightarrow & 1 \\ 100 & \longrightarrow & 0 \\ 011 & \longrightarrow & 0 \\ 010 & \longrightarrow & 0 \\ 001 & \longrightarrow & 1 \\ 000 & \longrightarrow & 0 \end{array}$$

The local rule replaces pattern 01 by pattern 10.

$$
\begin{array}{ccc}
111 & \longrightarrow & 1 \\
110 & \longrightarrow & 1 \\
101 & \longrightarrow & 1 \\
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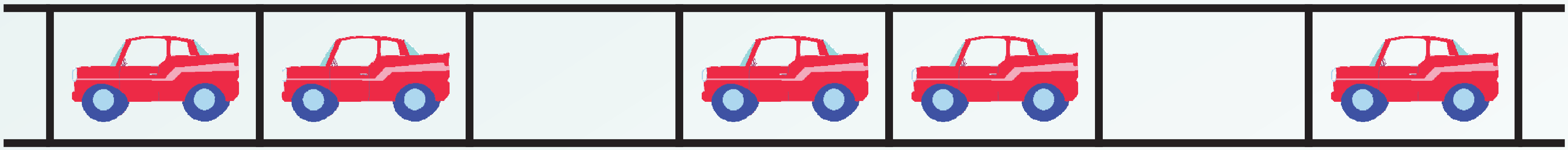

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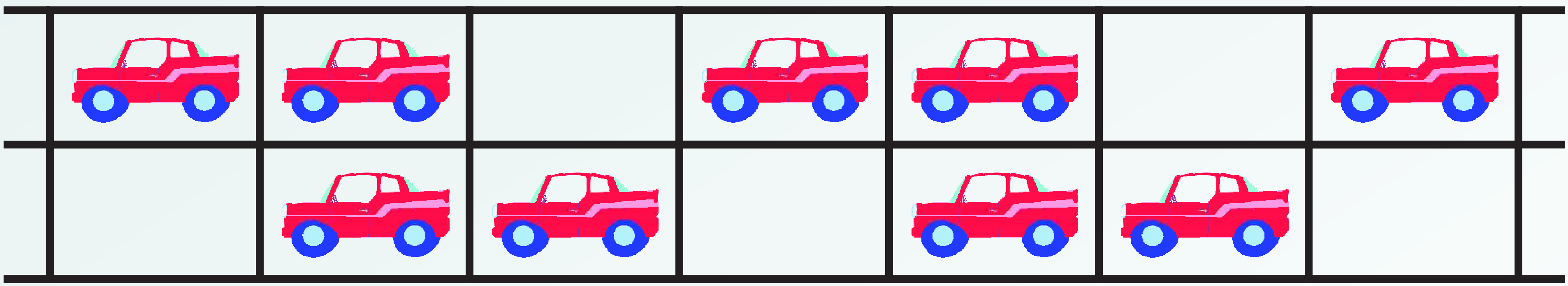

111	$\longrightarrow$	1
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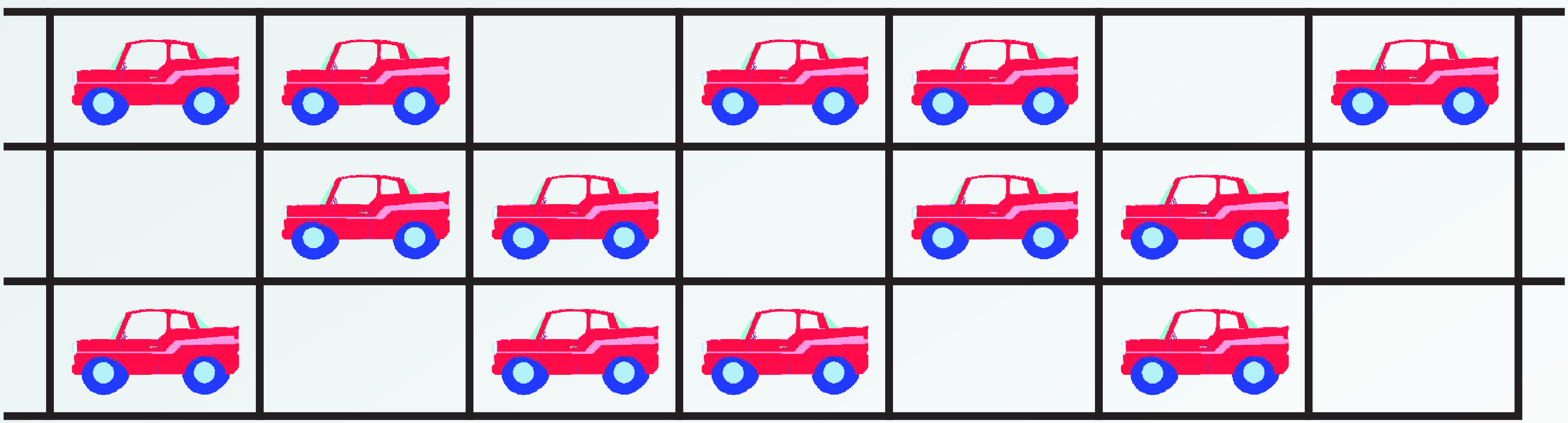

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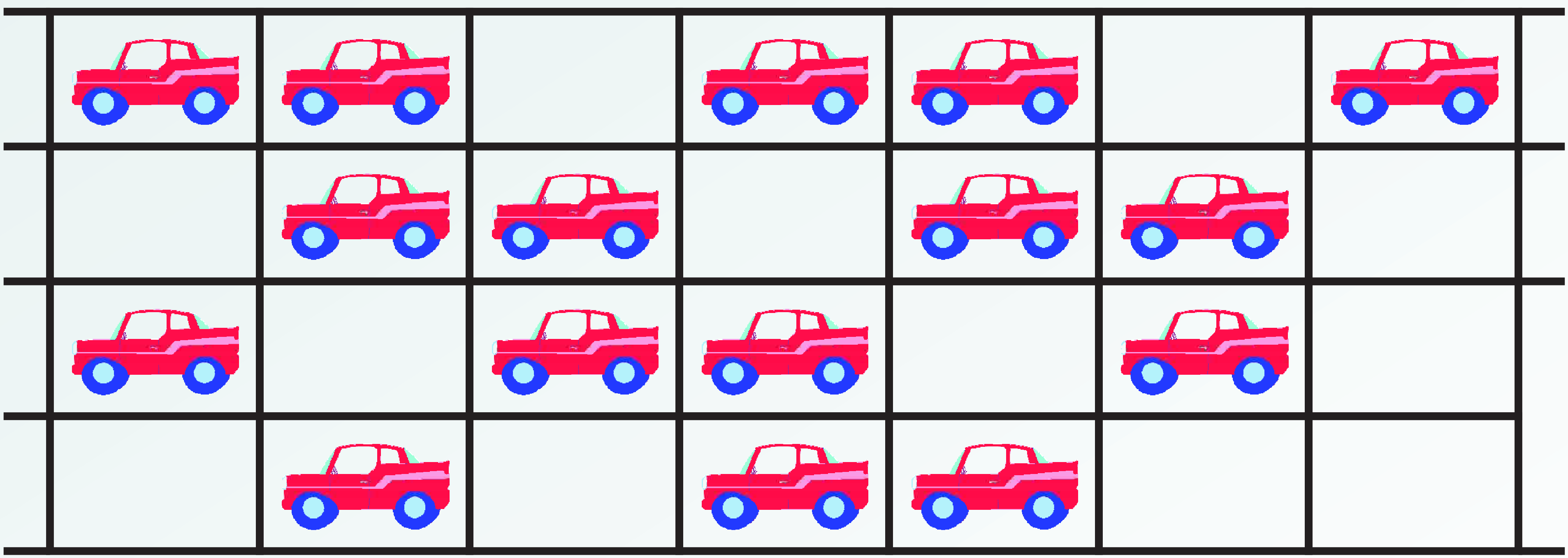
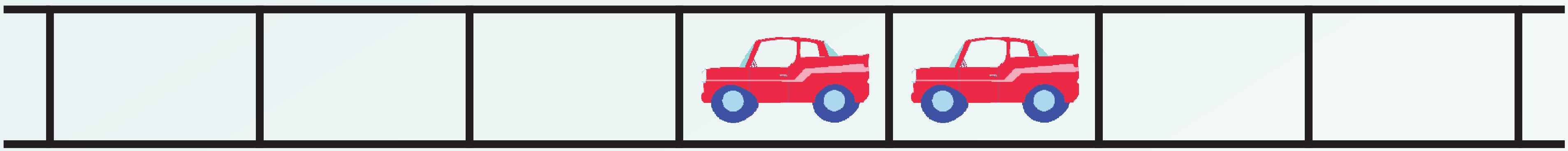
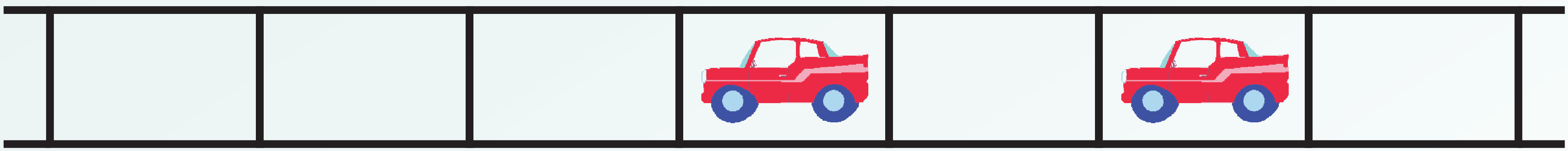


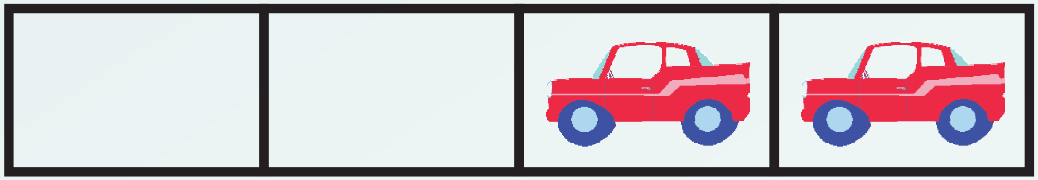
The local rule is balanced. However, there are two finite configurations with the same successor:

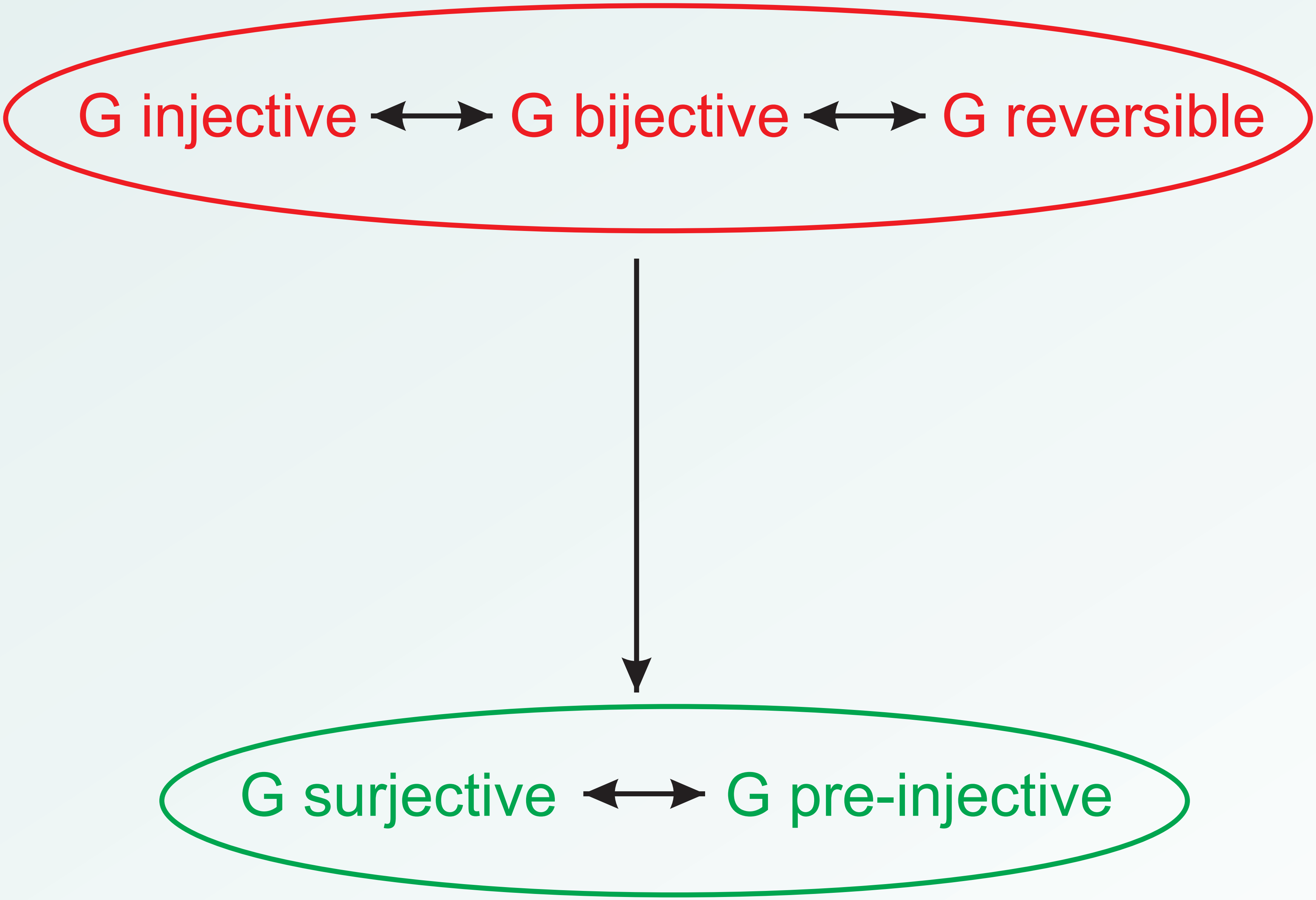


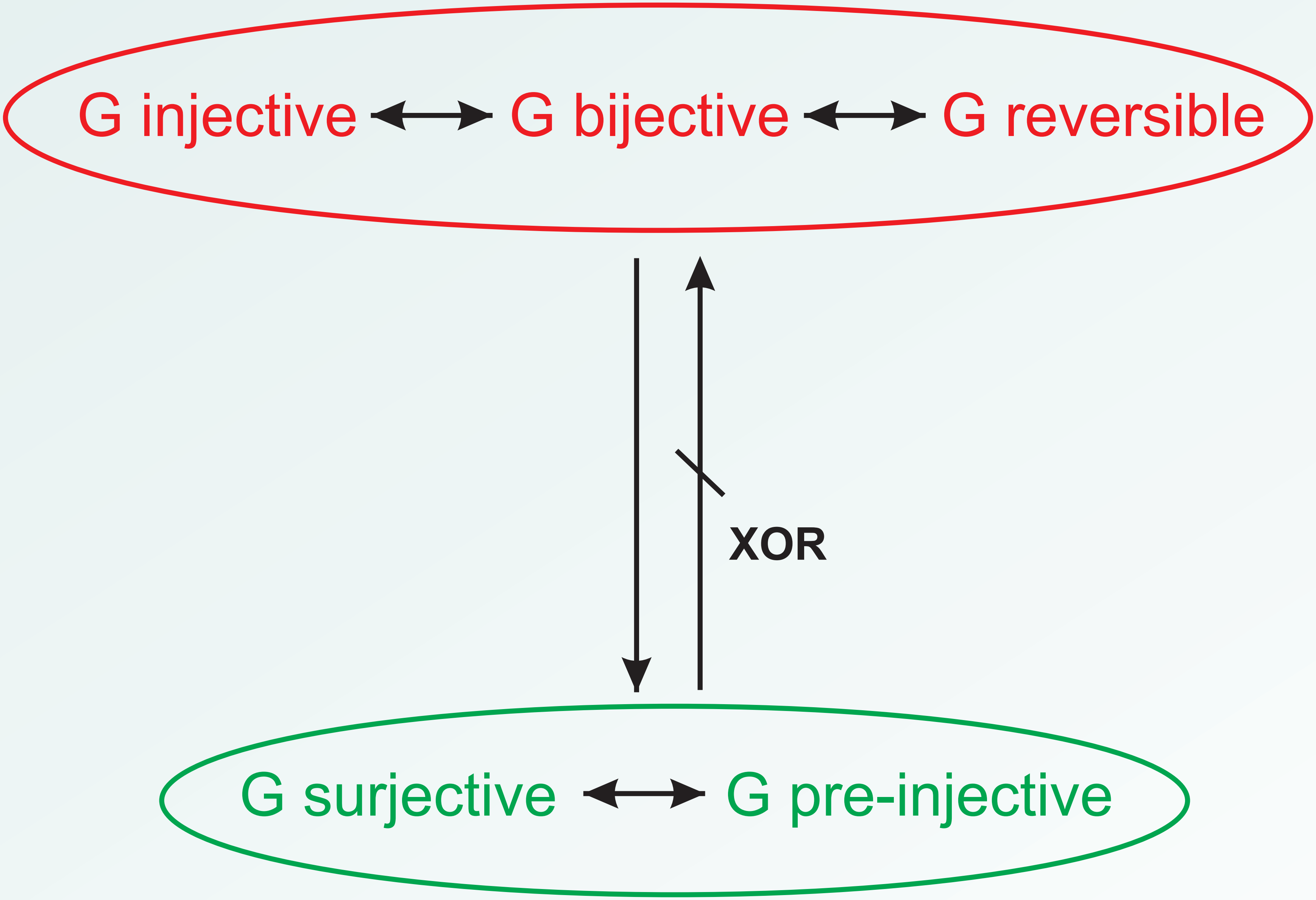


and hence traffic CA is not surjective.

There is an orphan of size four:




G injective
G bijective
G reversible
G surjective
G pre-injective


G injective
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G reversible
XOR
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G pre-injective

The xor-CA is the binary state CA with neighborhood $(0,1)$ and local rule

$$f(a,b) = a + b \pmod 2.$$

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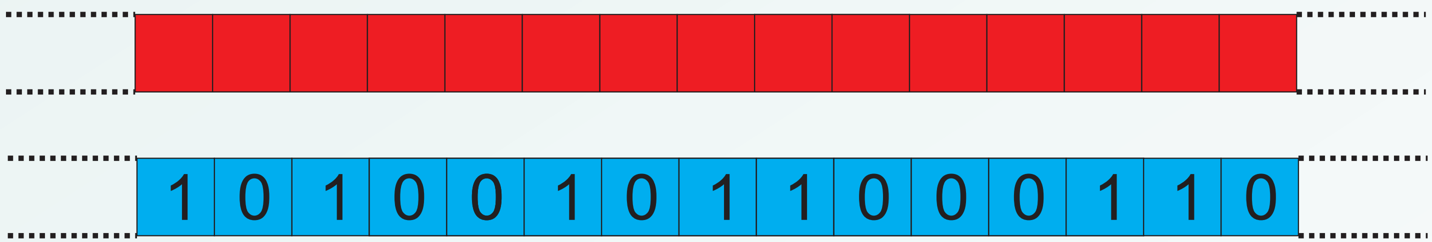
In the xor-CA every configuration has exactly two pre-images, so G is surjective but not injective:



One can freely choose one value in the pre-image, after which all remaining states are uniquely determined by the **left-permutativity** and the **right-permutativity** of xor.

The xor-CA is the binary state CA with neighborhood $(0, 1)$ and local rule

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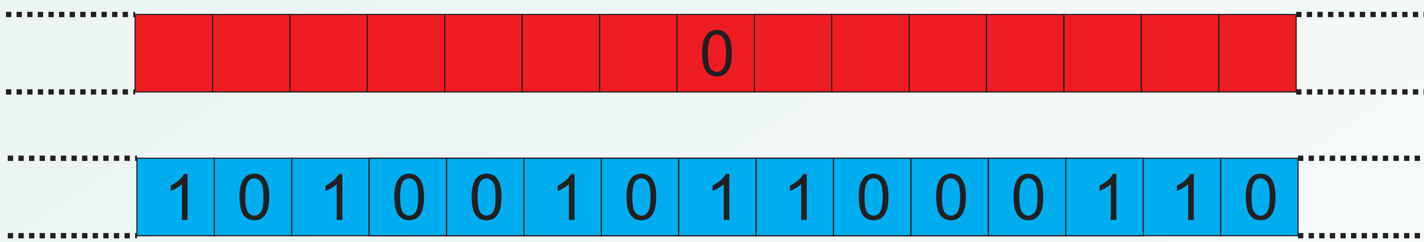

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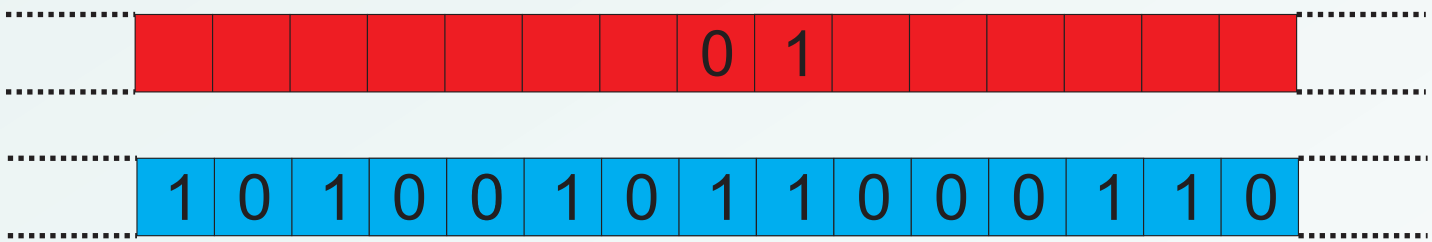

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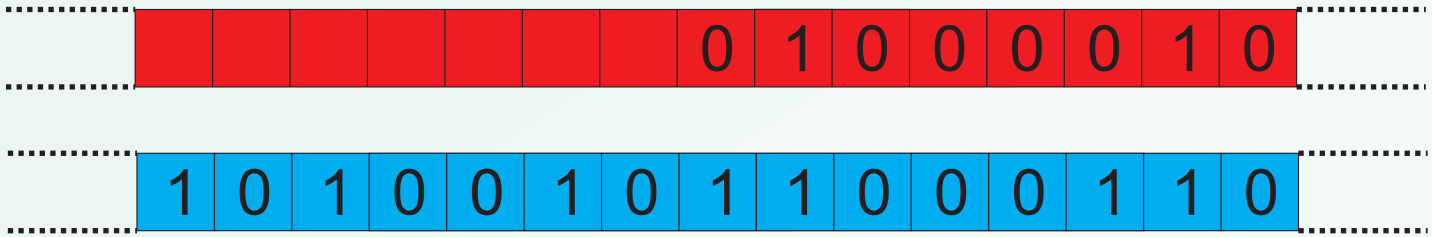

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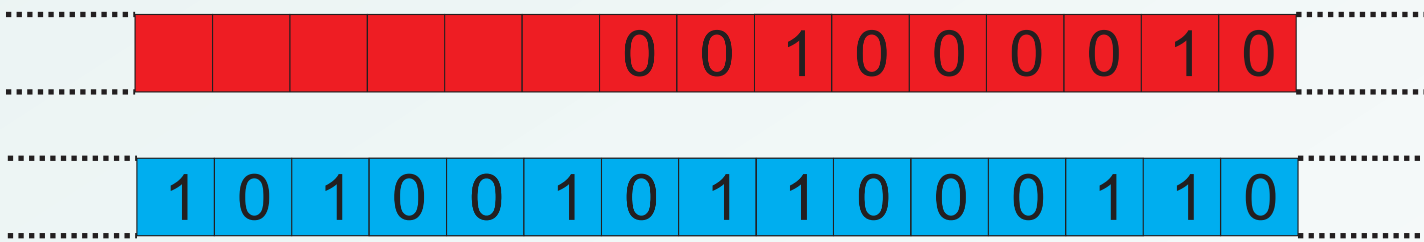

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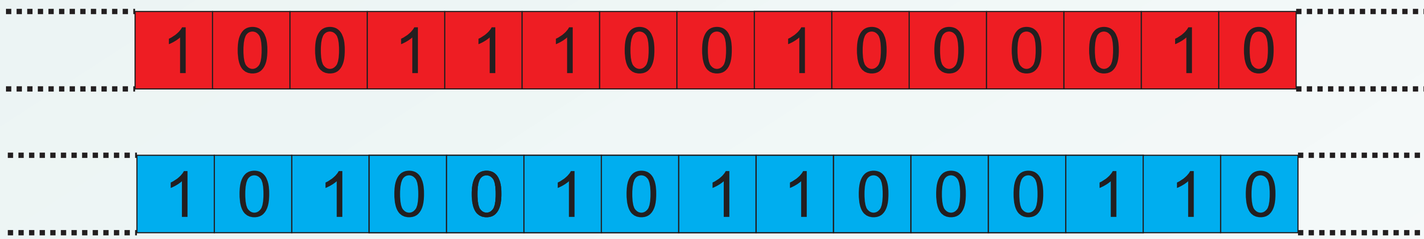

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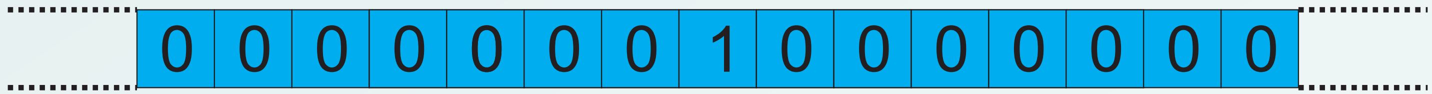
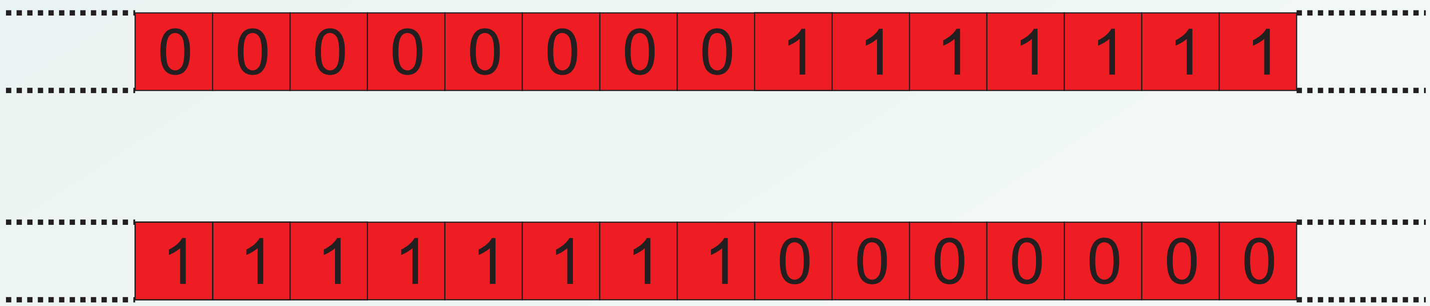
The two pre-images of the finite configuration



are both infinite:



So G is not surjective on 0-finite configurations.

Surjectivity and injectivity of G_P

Let G_P denote the restriction of cellular automaton G on (fully) periodic configurations.

Implications

$$\begin{aligned} \mathbf{G} \text{ injective} &\implies \mathbf{G_P} \text{ injective} \\ \mathbf{G_P} \text{ surjective} &\implies \mathbf{G} \text{ surjective} \end{aligned}$$

are easy. (Second one uses denseness of periodic configurations in $S^{\mathbb{Z}^d}$.)

We also have

$$\mathbf{G_P} \text{ injective } \implies \mathbf{G_P} \text{ surjective}$$

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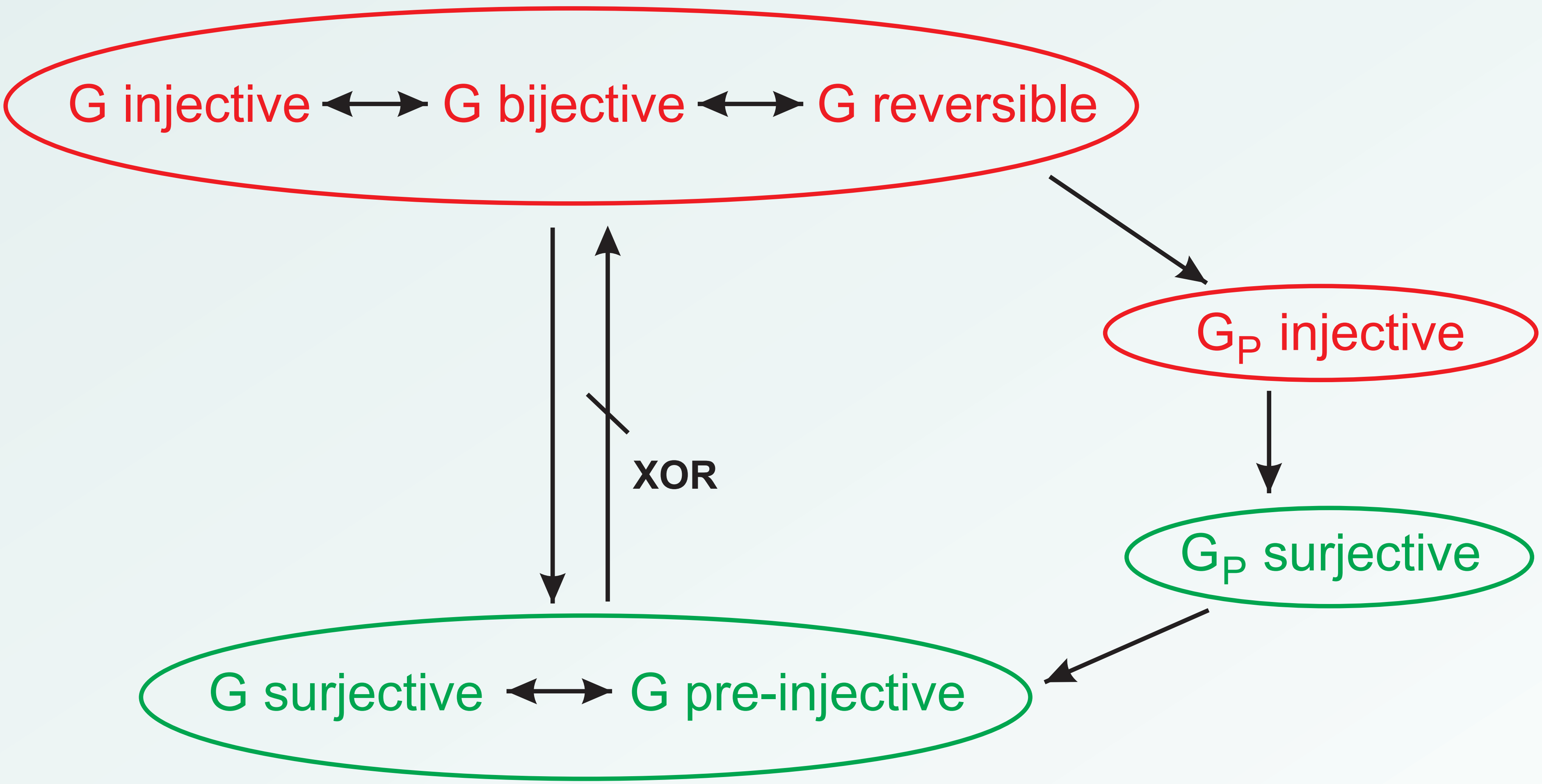
Indeed, fix any d linearly independent periods, and let $A \subseteq S^{\mathbb{Z}^d}$ be the set of configurations with these periods. Then

- A is finite,
- G is injective on A,
- $G(A) \subseteq A$.

We conclude that $G(A) = A$, and every periodic configuration has a periodic pre-image.



G injective
G bijective
G reversible
G_P injective
G_P surjective
XOR
G surjective
G pre-injective

Here we get the first **dimension sensitive** property. The following equivalences are only known to hold among one-dimensional CA:

$$\mathbf{G} \text{ injective} \iff \mathbf{G_P} \text{ injective}$$

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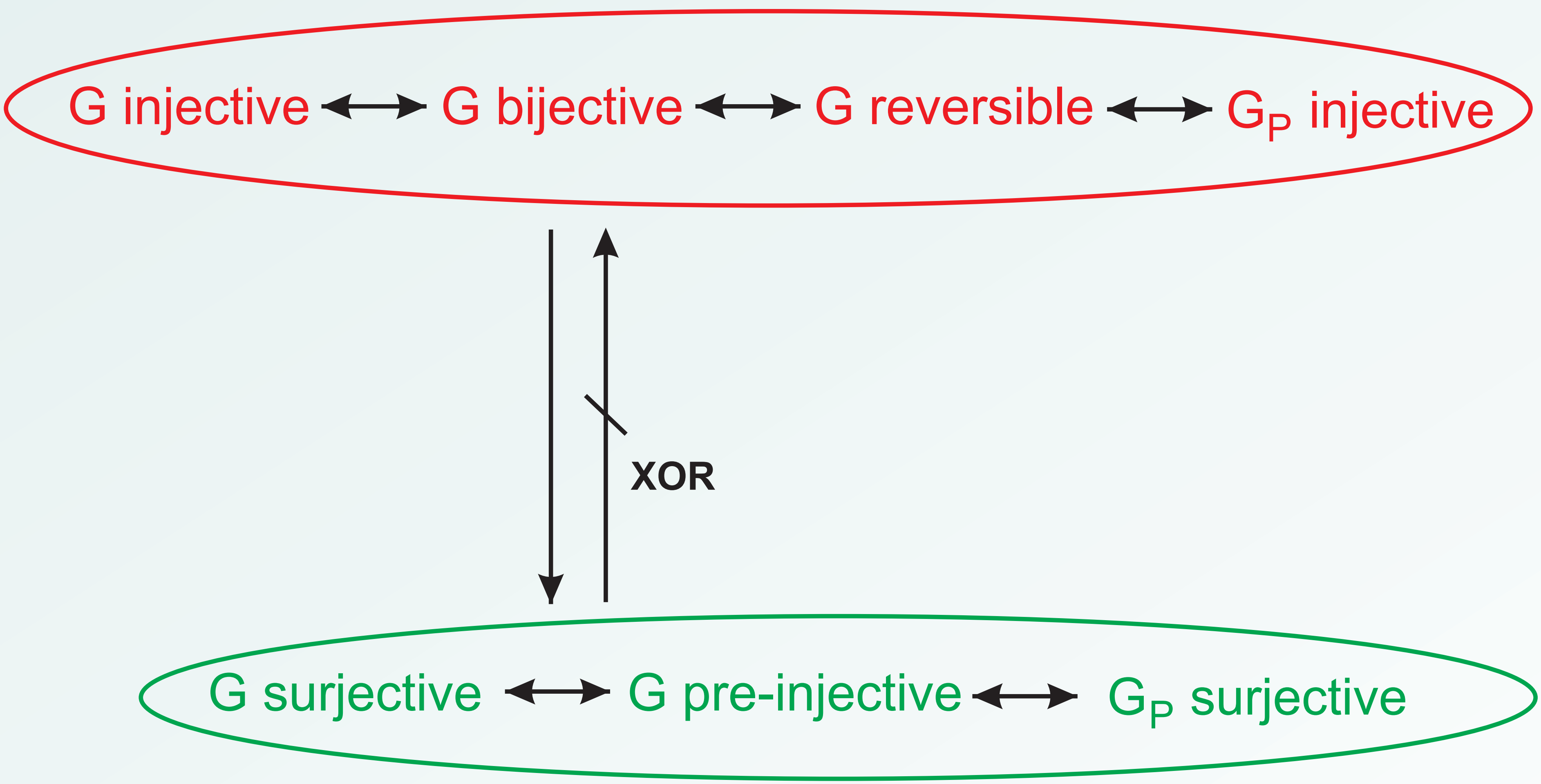
- The first equivalence is not true among two-dimensional CA: counter example Snake-XOR will be seen tomorrow.
- It is not known whether the second equivalence is true in 2D.

Only in 1D


G injective
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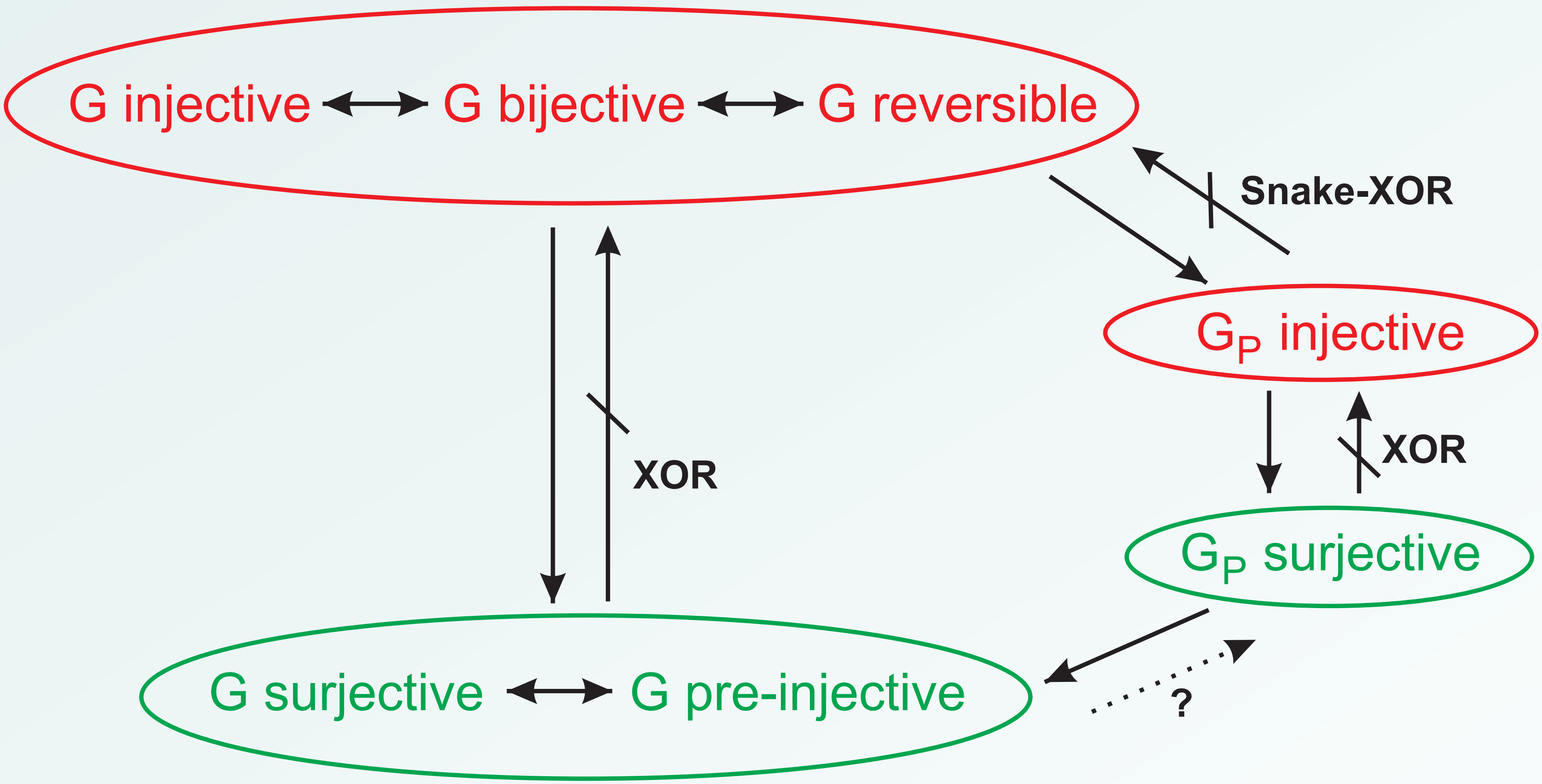
In 2D


G injective
G bijective
G reversible
Snake-XOR
G_P injective
XOR
XOR
G_P surjective
G surjective
G pre-injective
?

We have two proofs that injective CA are surjective:

$\mathbf{G}$ injective $\implies$ $\mathbf{G}$ pre-injective $\implies$ $\mathbf{G}$ surjective

$\mathbf{G}$ injective $\implies$ $\mathbf{G_P}$ injective $\implies$ $\mathbf{G_P}$ surjective $\implies$ $\mathbf{G}$ surjective

We have two proofs that injective CA are surjective:

$\mathbf{G}$ injective $\Longrightarrow$ $\mathbf{G}$ pre-injective $\Longrightarrow$ $\mathbf{G}$ surjective

$\mathbf{G}$ injective $\Longrightarrow$ $\mathbf{G_P}$ injective $\Longrightarrow$ $\mathbf{G_P}$ surjective $\Longrightarrow$ $\mathbf{G}$ surjective

It is good to have both implication chains available, if one wants to generalize results to cellular automata whose underlying grid is not $\mathbb{Z}^d$ but some other group.

- The first chain generalizes to all **amenable** groups.
- The second chain generalizes to **residually finite** groups.

A group is called **surjunctive** if every injective CA on the group is also surjective. It is not known if all groups are surjunctive.