Lecture 3: Tilings and undecidability

- Wang tiles and the tiling problem
- A (relatively) small aperiodic tile set
- Undecidability of the tiling problem

Wang tiles and decidability questions

Suppose we are given a cellular automaton and want to know if it is reversible or surjective ? Is there an algorithm to decide this ? Or can we determine if the dynamics of a given CA is trivial in the sense that after a while all activity has died ?

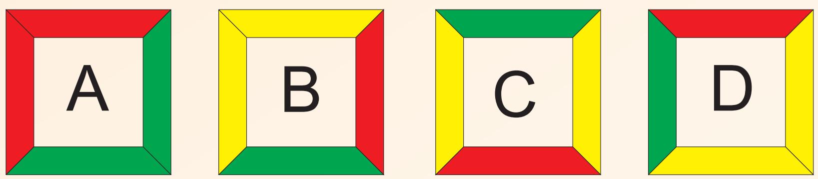
It turns out that many such algorithmic problems are **undecidable**. In some cases there is an algorithm for one-dimensional CA while the two-dimensional case is undecidable.

A useful tool to obtain undecidability results is the concept of **Wang tiles** and the undecidable **tiling problem**.

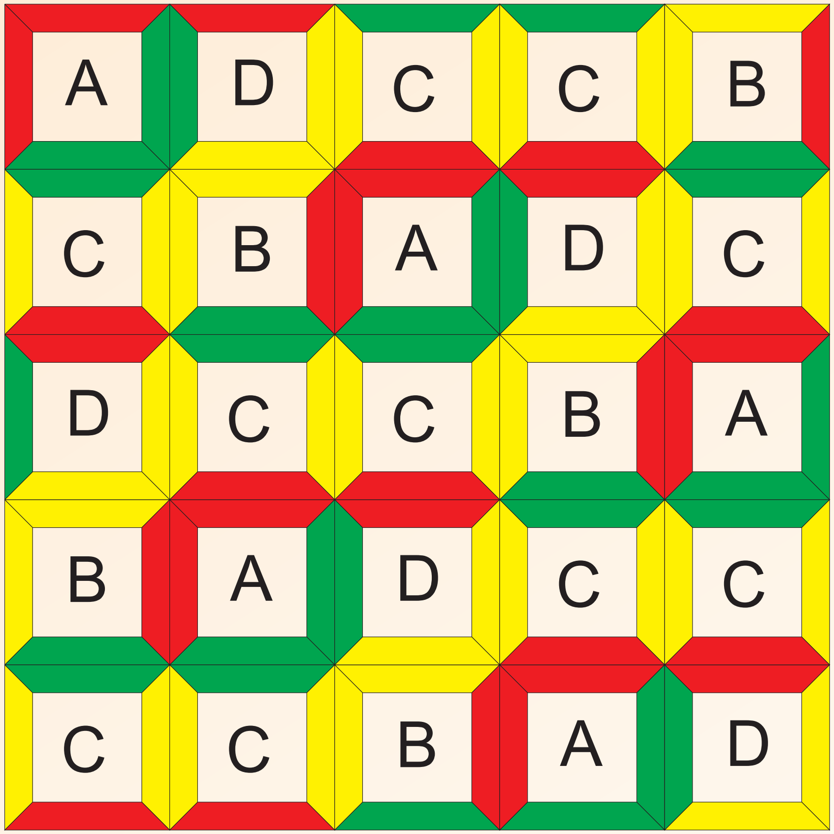
A **Wang tile** is a unit square tile with colored edges. A tile set T is a finite collection of such tiles. A valid tiling is an assignment

$$\mathbb{Z}^2 \longrightarrow T$$

of tiles on infinite square lattice so that the abutting edges of adjacent tiles have the same color.



With copies of the given four tiles we can properly tile a 5×5 square...



... and since the colors on the borders match this square can be repeated to form a valid periodic tiling of the plane.

The **tiling problem** (or the **Domino problem**) of Wang tiles is the decision problem to determine if a given finite set of Wang tiles admits a valid tiling of the plane.

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Follows from (1): Just try tiling larger and larger squares until (if ever) a square is found that can not be tiled.

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Reason: Just try tiling rectangles until (if ever) a valid tiling is found where colors on the top and the bottom match, and left and the right sides match as well.

(1) If T admits valid tilings inside squares of arbitrary size then it admits a valid tiling of the whole plane.

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Execute semi-algorithms (2) and (3) in parallel:

- If T does not tile the plane, (2) will eventually halt.
- If T admits a periodic tiling, (3) will eventually halt.

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Is this an algorithm that solves the **tiling problem** ?

No! There are tile sets that fall between cases (2) and (3). They admit valid tilings but do not admit any periodic tilings.

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Berger in fact proved more:

Theorem (R.Berger 1966): The tiling problem of Wang tiles is undecidable.

A small aperiodic tile set

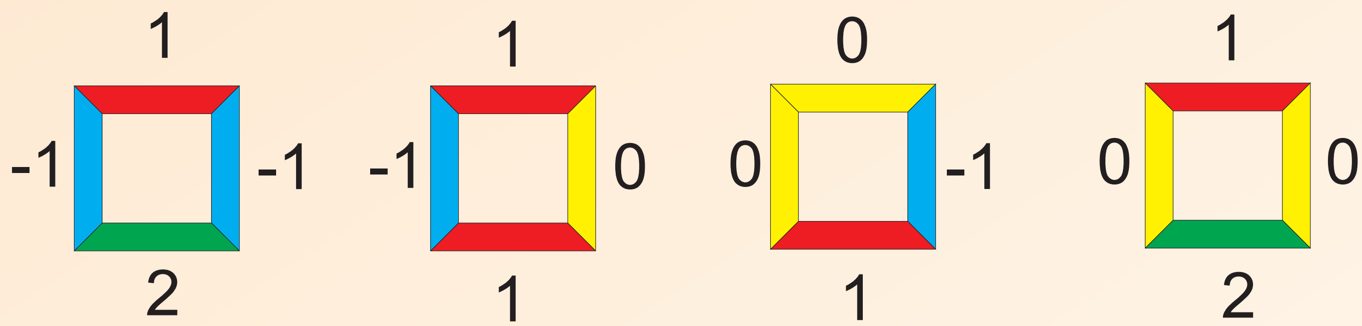
We construct a 14 tile set that simulates piecewise linear dynamical systems.

Non-periodic systems $\Longrightarrow$ aperiodic tile sets.

Immortality of such dynamics is undecidable

$\Longrightarrow$ the tiling problem is undecidable.

The colors in our Wang tiles are real numbers, for example



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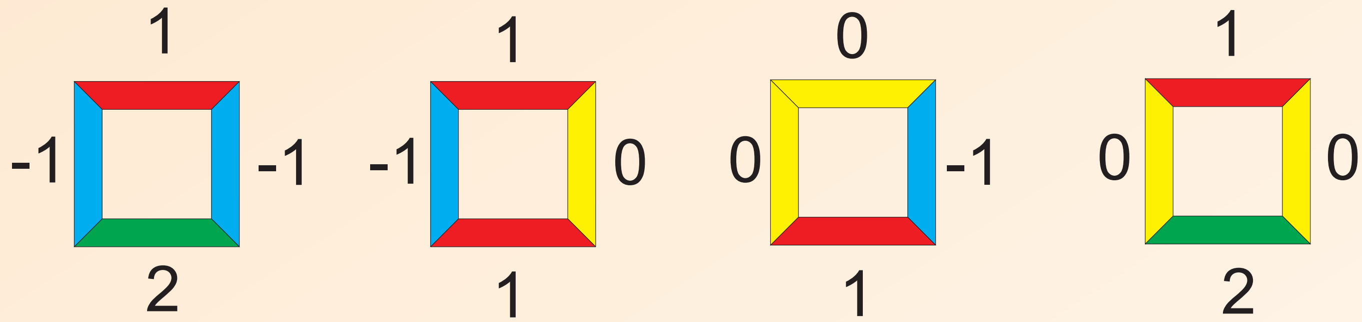
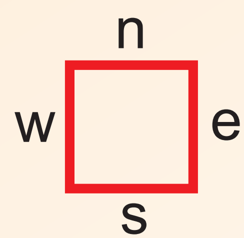


We say that tile



multiplies by number $q \in \mathbb{R}$ if

$$qn + w = s + e.$$

(The **"Input"** n comes from the north, and the **"carry in"** w from the west is added to the product qn. The result is split between the **"output"** s to the south and the **"carry out"** e to the east.)

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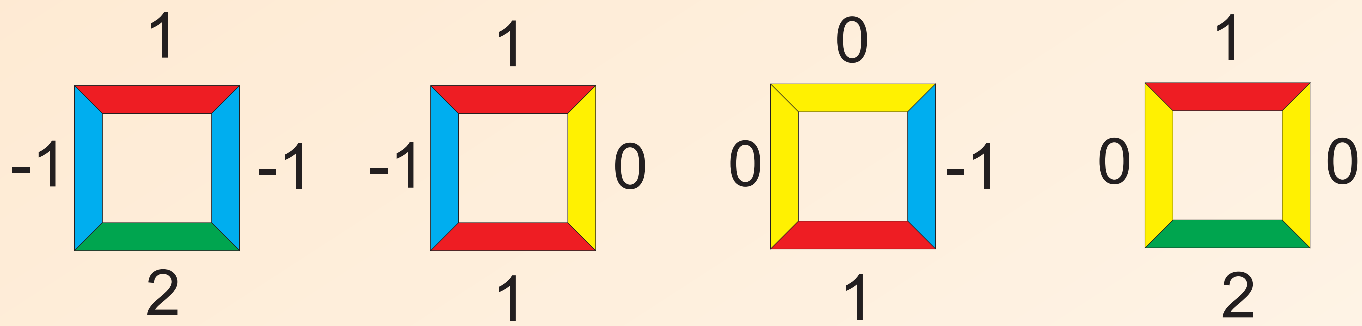

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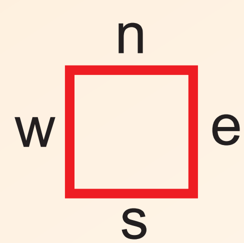
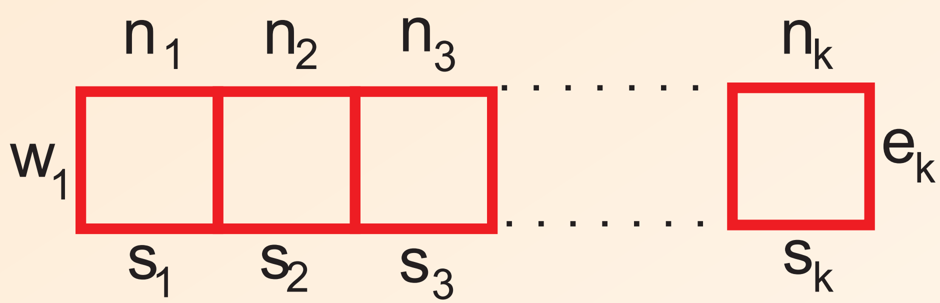


multiplies by number $q \in \mathbb{R}$ if

$$qn + w = s + e.$$

The four sample tiles above all multiply by $q = 2$.

Suppose we have a correctly tiled horizontal segment where all tiles multiply by the same q.



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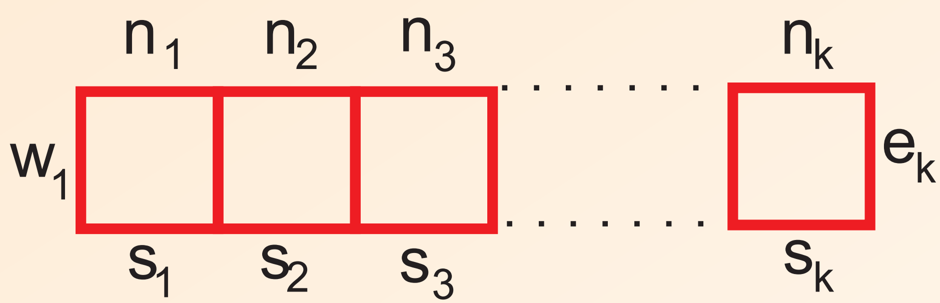

It easily follows that

$$q(n_1 + n_2 + \ldots + n_k) + w_1 = (s_1 + s_2 + \ldots + s_k) + e_k.$$

To see this, simply sum up the equations

$$\begin{aligned} qn_1 + w_1 &= s_1 + e_1 \\ qn_2 + w_2 &= s_2 + e_2 \\ &\vdots \\ qn_k + w_k &= s_k + e_k, \end{aligned}$$

taking into account that always $e_i = w_{i+1}$.

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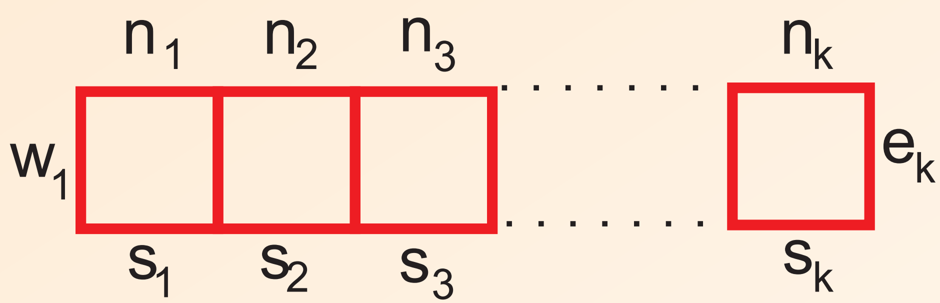
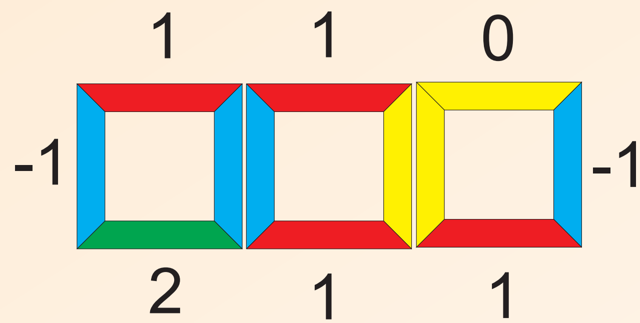


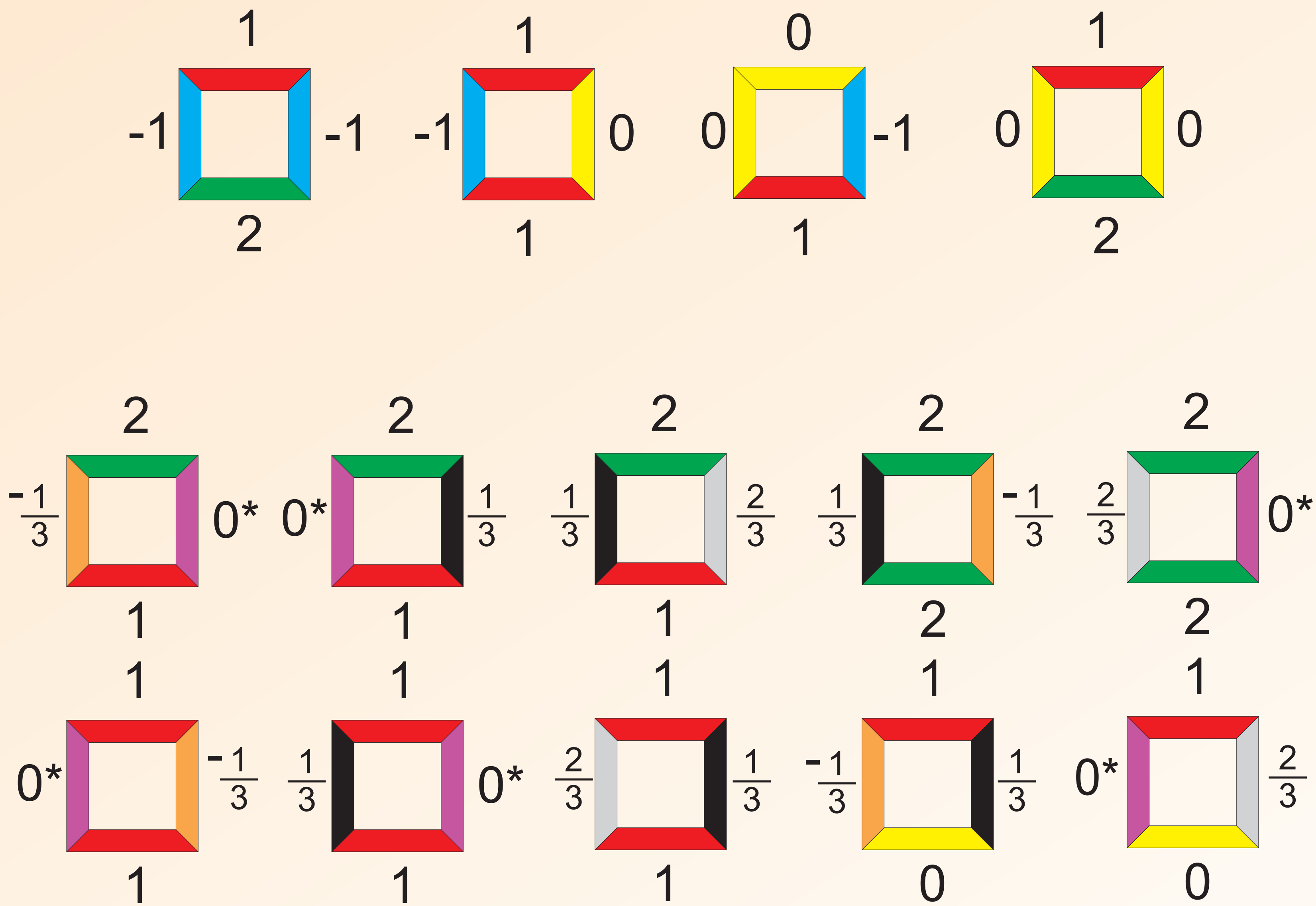
If, moreover, the segment begins and ends in the same color $(w_1 = e_k)$ then

$$q(n_1 + n_2 + \ldots + n_k) = (s_1 + s_2 + \ldots + s_k).$$

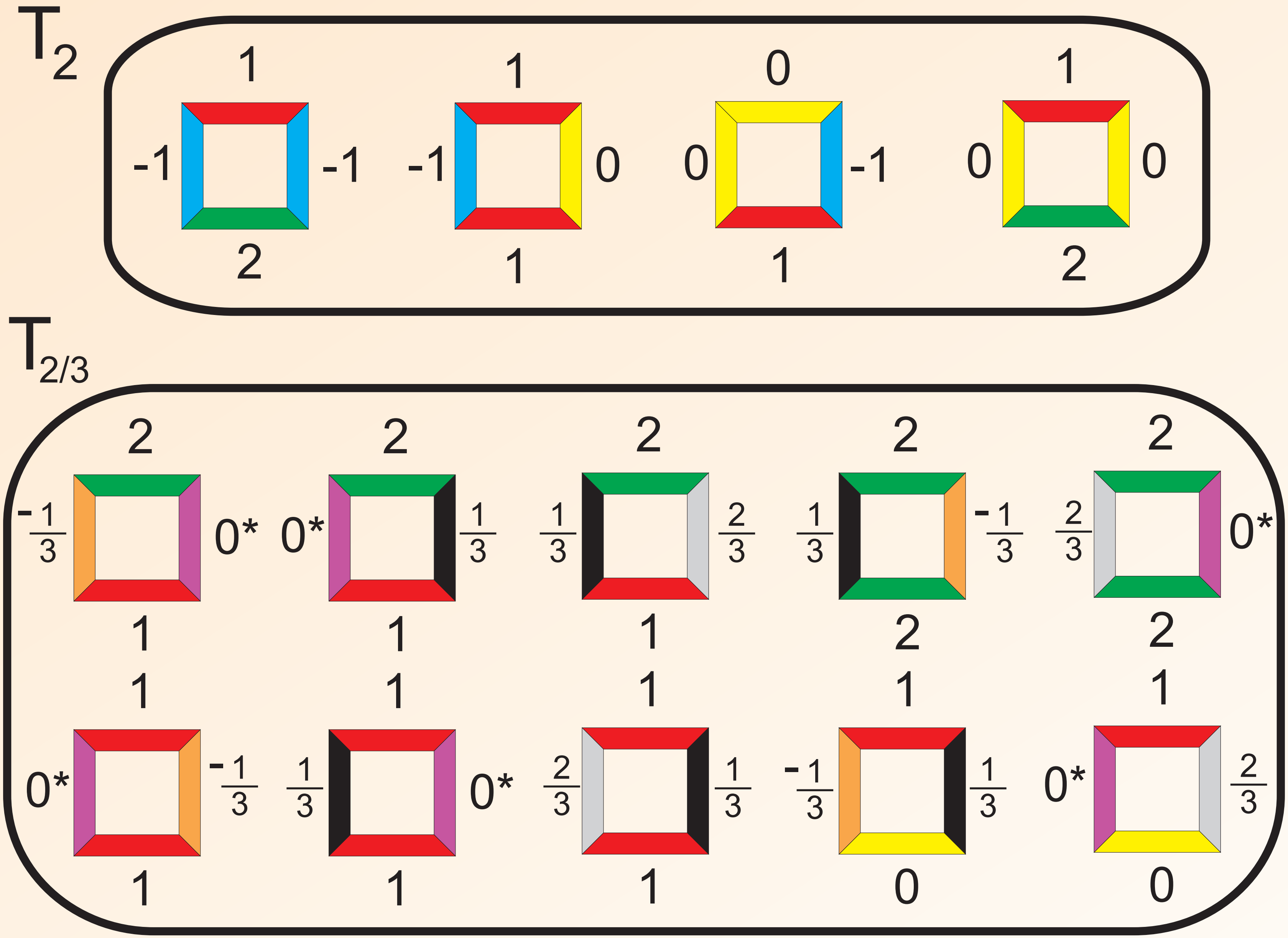
For example, using our three sample tiles that multiply by $q = 2$ we can form the segment



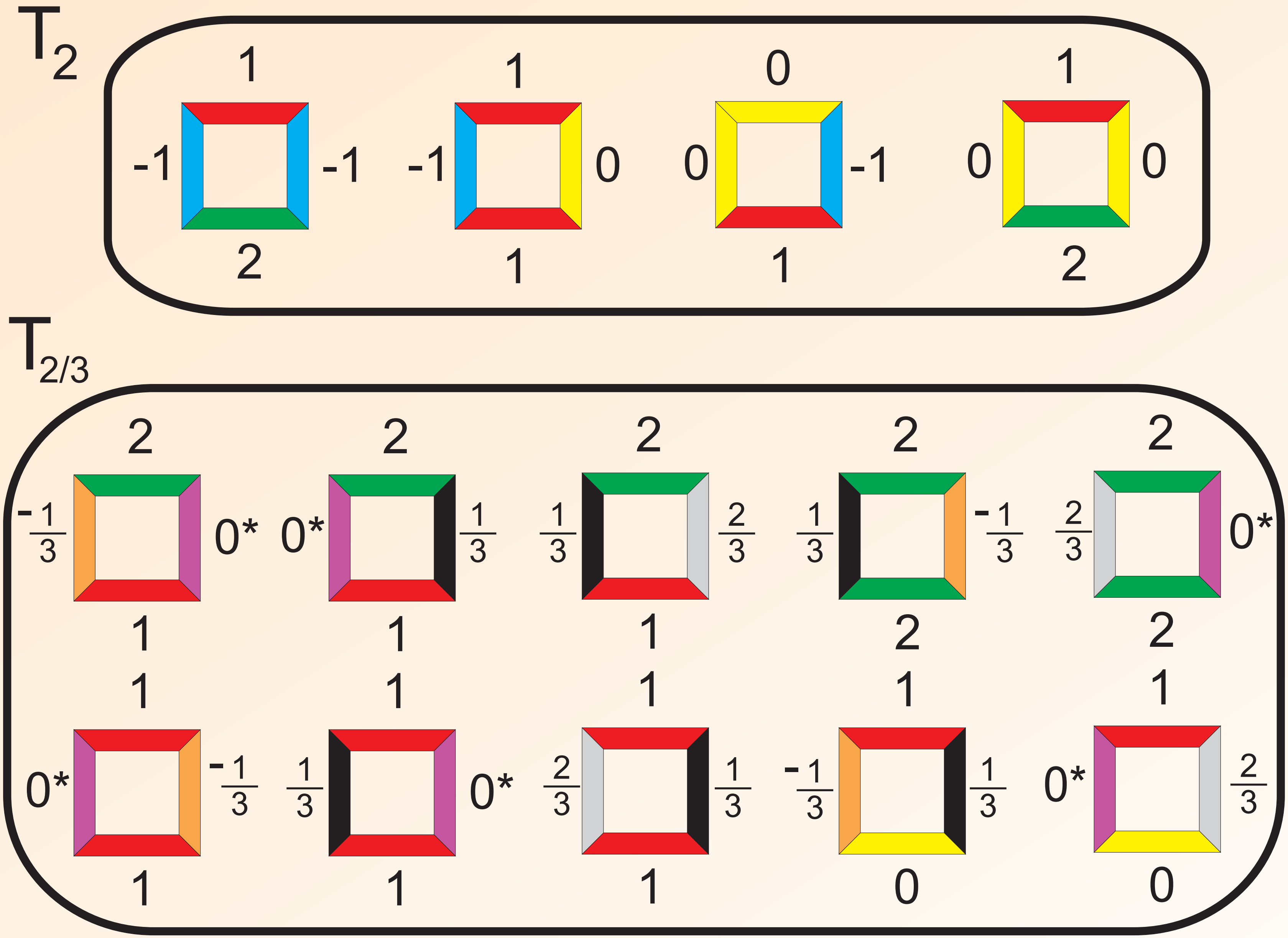
in which the sum of the bottom labels is twice the sum of the top labels.



Our aperiodic tile set consists of the four tiles that multiply by 2, together with a family of ten tiles that all multiply by $\frac{2}{3}$.



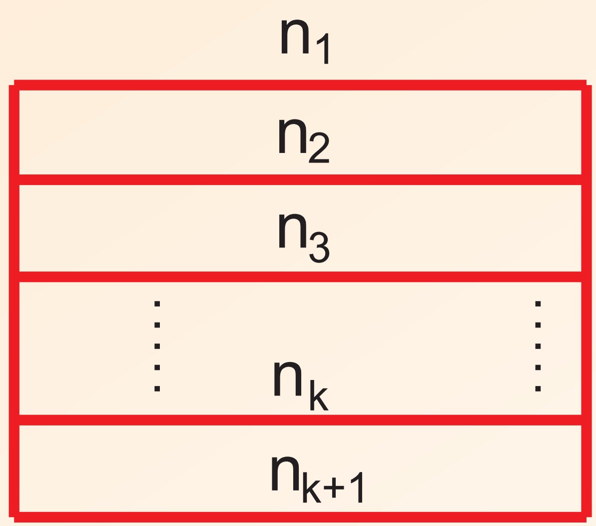
Let us call these two tile sets T_2 and $T_{2/3}$.



Vertical edge colors of the two parts are made disjoint, so any properly tiled horizontal row comes entirely from one of the two sets.

No periodic tiling exists.

Suppose the contrary: A rectangle can be tiled whose top and bottom rows match and left and right sides match.



Denote by n_i the sum of the numbers on the i'th horizontal row. Let the tiles of the i'th row multiply by $q_i \in \{2, \frac{2}{3}\}$.

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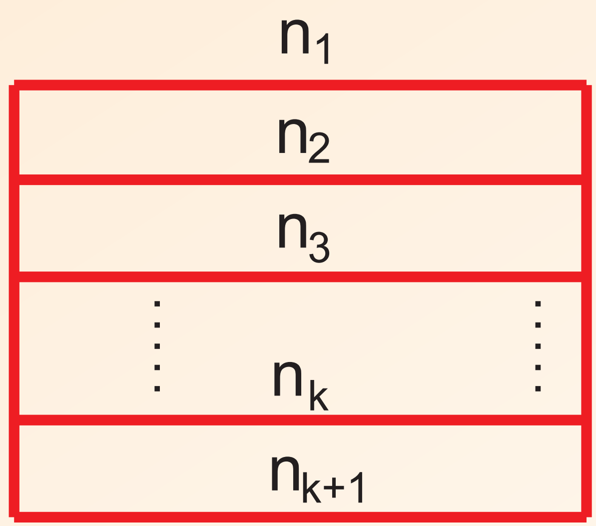

Denote by n_i the sum of the numbers on the i'th horizontal row. Let the tiles of the i'th row multiply by $q_i \in \{2, \frac{2}{3}\}$.

Then $n_{i+1} = q_i n_i$, for all i.

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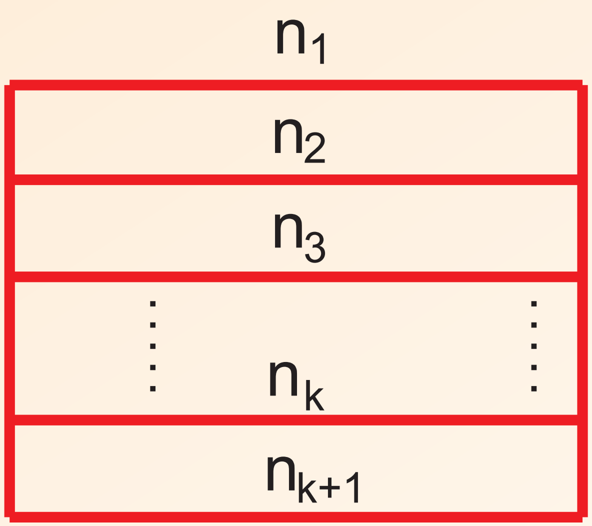

So we have $q_1 q_2 q_3 \ldots q_k n_1 = n_{k+1}$

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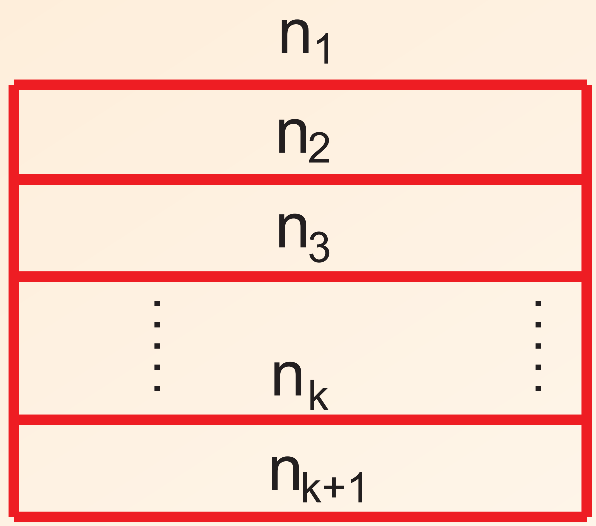

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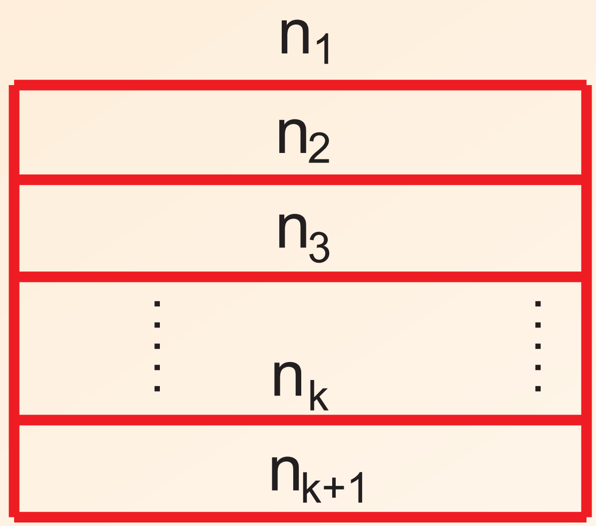

So we have $q_1 q_2 q_3 \ldots q_k n_1 = n_{k+1} = n_1$.

Clearly $n_1 > 0$, so we have $q_1 q_2 q_3 \ldots q_k = 1$.

But this is not possible since 2 and 3 are relatively prime: No product of numbers 2 and $\frac{2}{3}$ can equal 1.

Next step: Proof that a valid tiling of the plane exists.

We use **sturmian** or **balanced** representations of real numbers as bi-infinite sequences of two closest integers.

The representation of any $\alpha \in \mathbb{R}$ is the sequence $B(\alpha)$ whose k'th element is

$$B_k(\alpha) = \lfloor k\alpha \rfloor - \lfloor (k-1)\alpha \rfloor.$$

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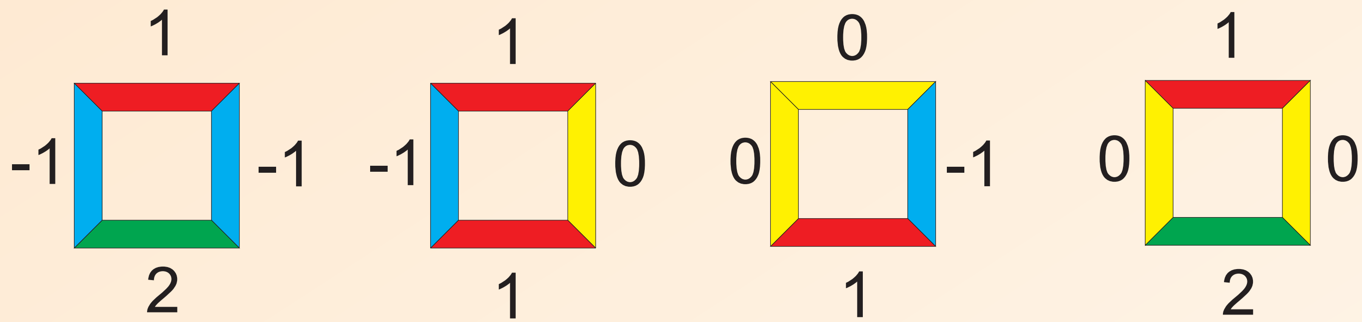
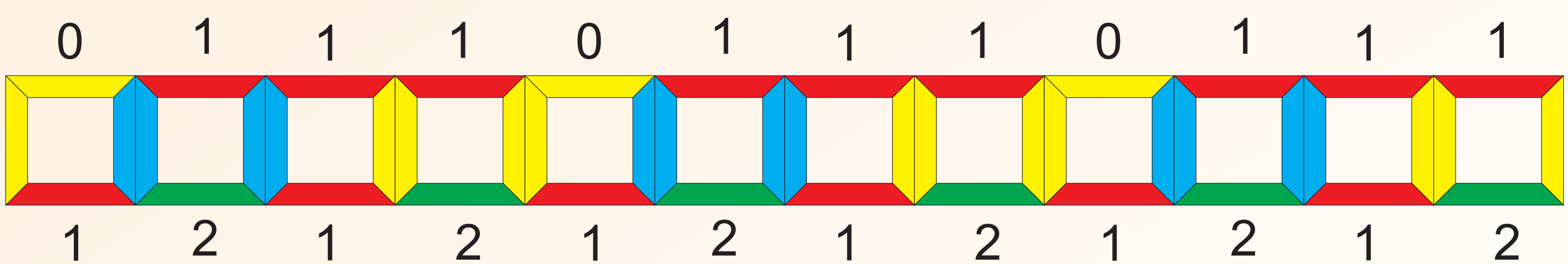
B(\sqrt{2}) &= \ldots 1\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 1\ 2\ 1\ 1\ \ldots
\end{aligned}$$

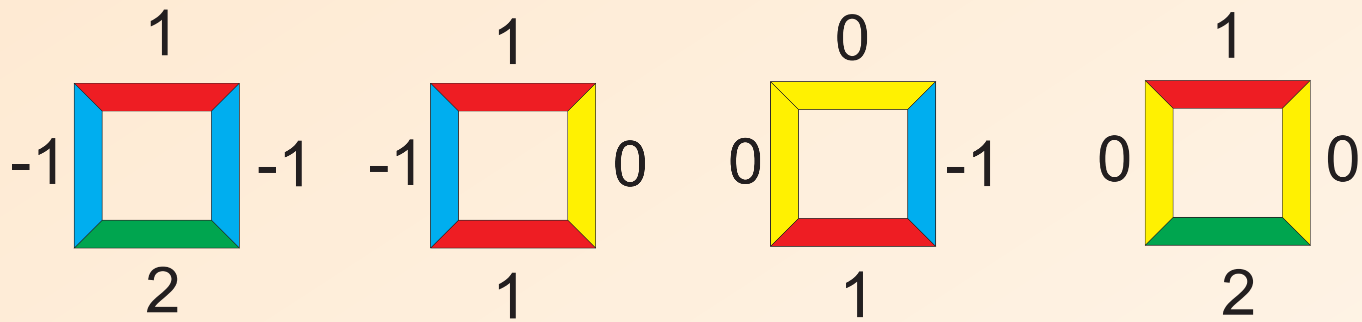


The first tile set T_2 admits a tiling of every infinite horizontal strip whose top and bottom labels read $B(\alpha)$ and $B(2\alpha)$, for all $\alpha \in \mathbb{R}$ satisfying

$$\begin{array}{ccccl} 0 & \leq & \alpha & \leq & 1, \text{ and} \\ 1 & \leq & 2\alpha & \leq & 2. \end{array}$$

For example, with $\alpha = \frac{3}{4}$:





The first tile set T_2 admits a tiling of every infinite horizontal strip whose top and bottom labels read $B(\alpha)$ and $B(2\alpha)$, for all $\alpha \in \mathbb{R}$ satisfying

$$\left.\begin{array}{ccccc} 0 & \leq & \alpha & \leq & 1, \text{ and} \\ 1 & \leq & 2\alpha & \leq & 2. \end{array}\right\} \iff \frac{1}{2} \leq \alpha \leq 1$$

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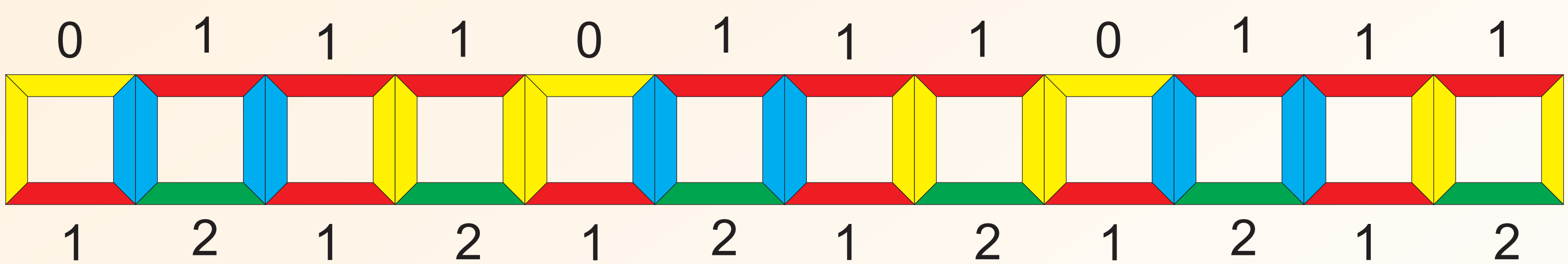
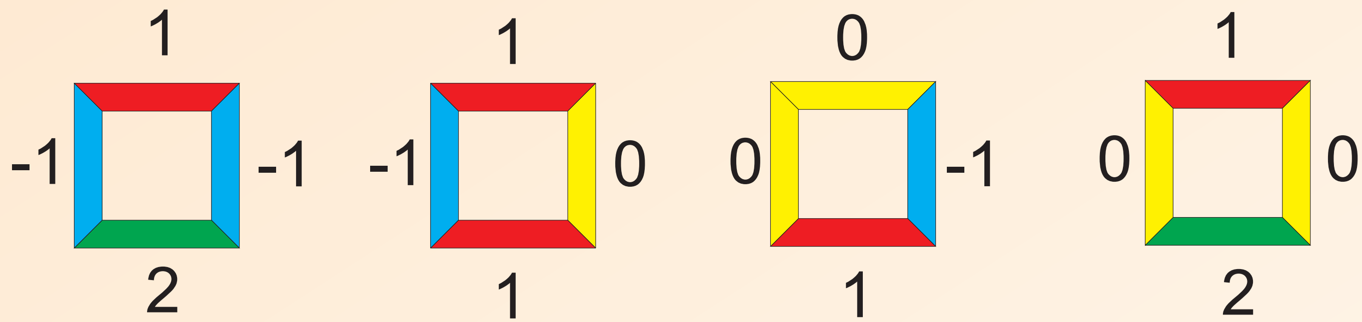
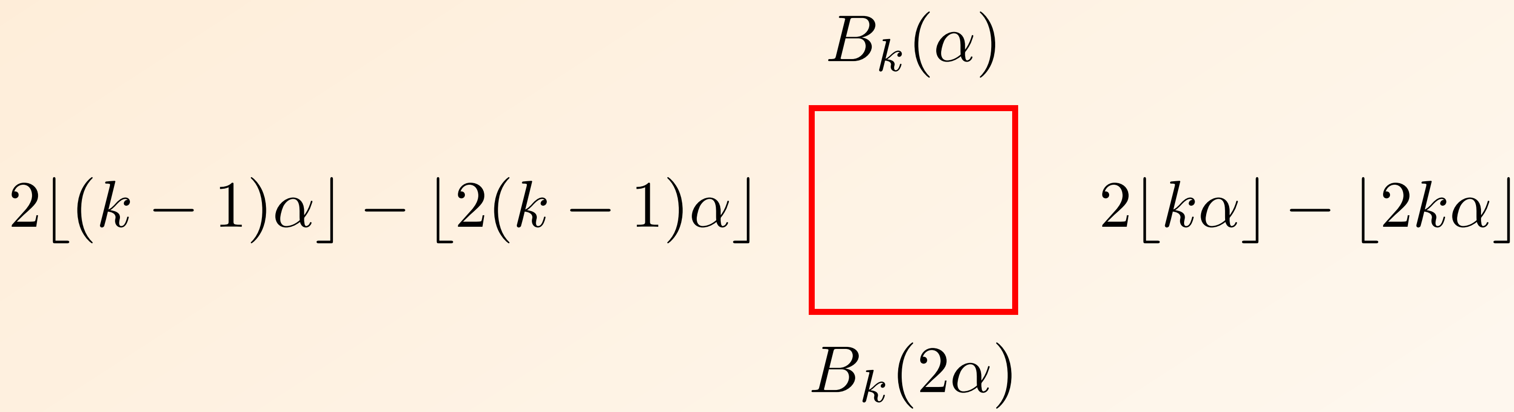
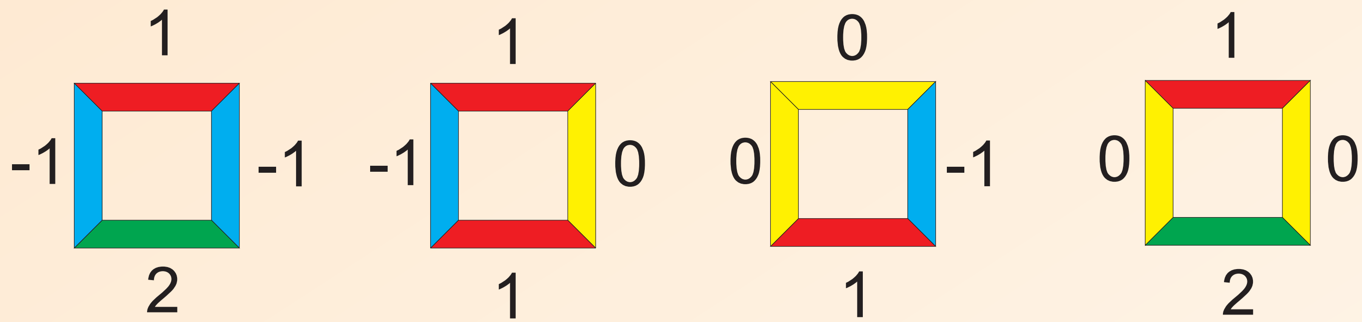




This is guaranteed by including in the tile set for every $\frac{1}{2} \leq \alpha \leq 1$ and every $k \in \mathbb{Z}$ the following tile





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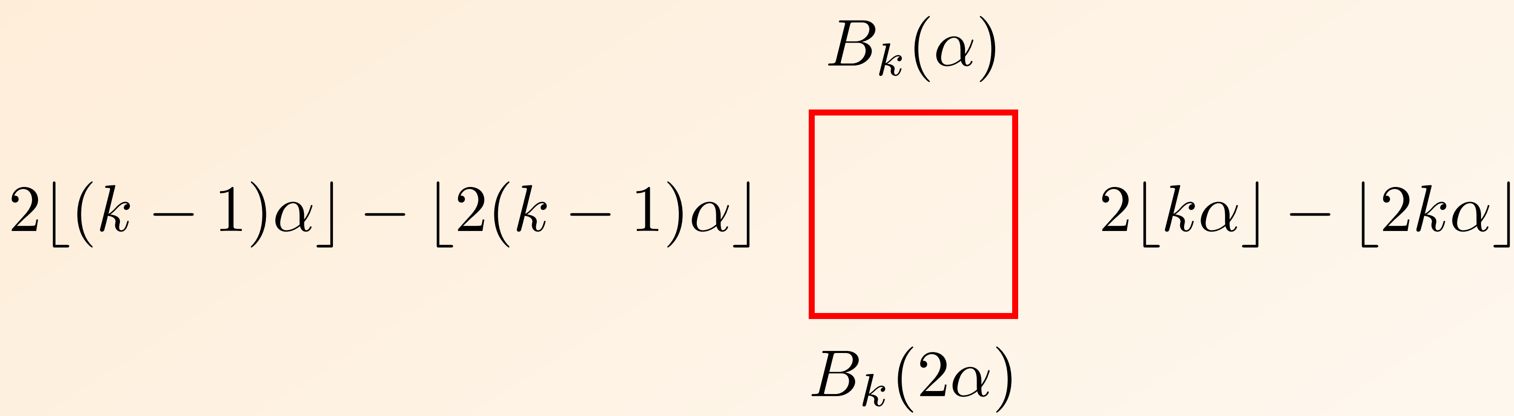
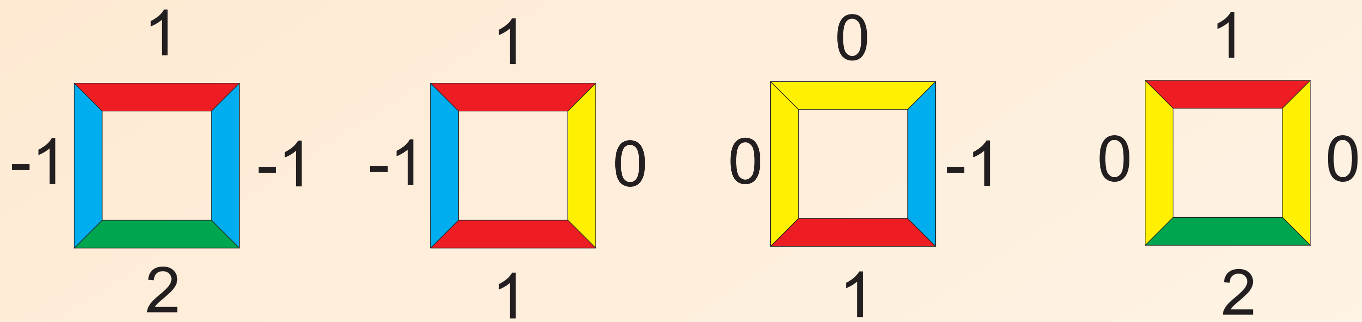


(1) For fixed α the tiles for consecutive $k \in \mathbb{Z}$ match so that a horizontal row can be formed whose top and bottom labels read the balanced representations of α and 2α, respectively.



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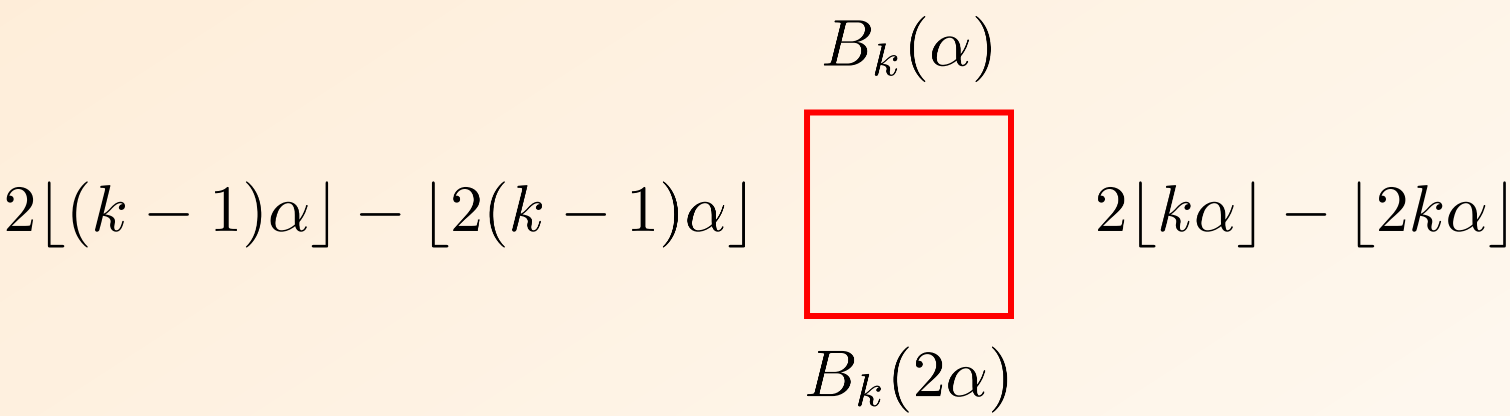
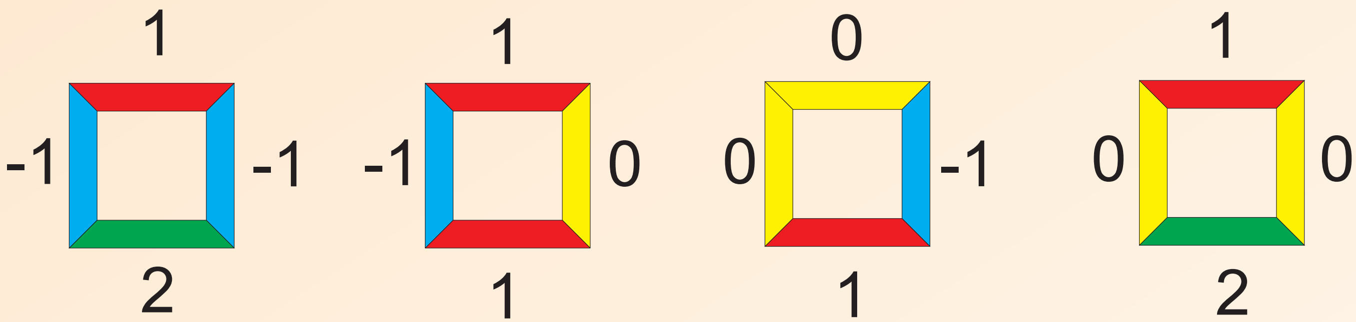


(2) A direct calculation shows that the tile multiplies by 2, that is,

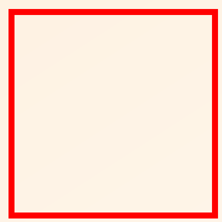
$$2n + w = s + e.$$



This is guaranteed by including in the tile set for every $\frac{1}{2} \leq \alpha \leq 1$ and every $k \in \mathbb{Z}$ the following tile

$$B_k(\alpha)$$

$$2\lfloor (k-1)\alpha \rfloor - \lfloor 2(k-1)\alpha \rfloor$$



$$2\lfloor k\alpha \rfloor - \lfloor 2k\alpha \rfloor$$

$$B_k(2\alpha)$$

(3) There are only finitely many such tiles, even though there are infinitely many $k \in \mathbb{Z}$ and α. The tiles are four tiles of T_2.

$$B_k(\alpha)$$

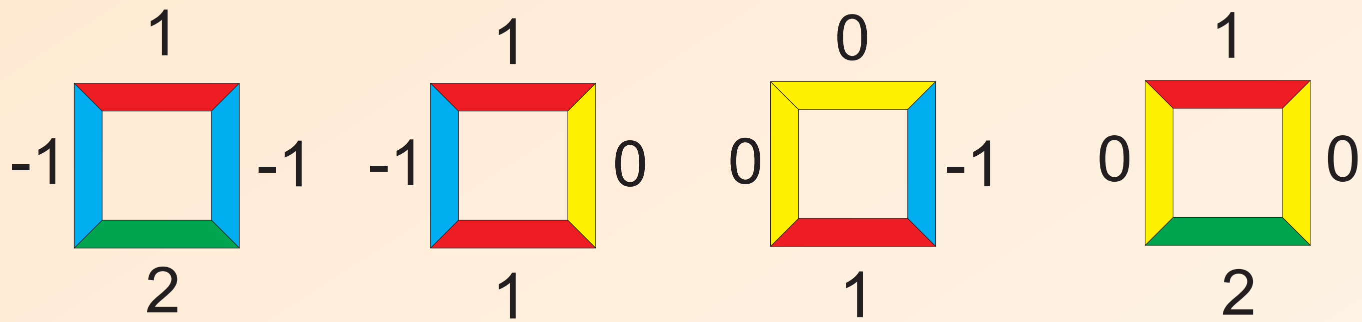
$$2\lfloor (k-1)\alpha \rfloor - \lfloor 2(k-1)\alpha \rfloor \qquad \boxed{} \qquad 2\lfloor k\alpha \rfloor - \lfloor 2k\alpha \rfloor$$

$$B_k(2\alpha)$$

There are finitely many such tiles because the vertical label is an integer satisfying

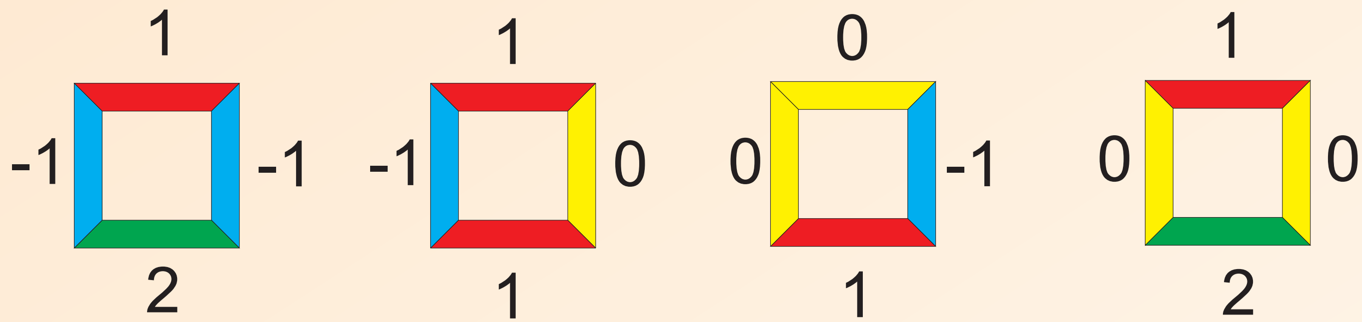
$$-2 \leq 2(k\alpha - 1) - 2k\alpha < \mathbf{2}\lfloor \mathbf{k}\alpha \rfloor - \lfloor \mathbf{2k}\alpha \rfloor < 2k\alpha - (2k\alpha - 1) = 1,$$

i.e., it is -1 or 0 for all α and k.

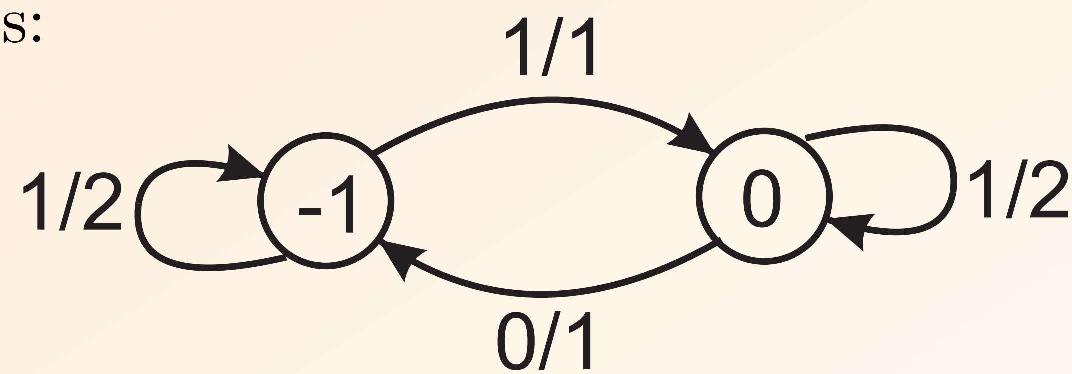


Our tile set T_2 simply contains all tiles that multiply by 2 among the tiles whose

- top label $\in \{0, 1\}$,
- bottom label $\in \{1, 2\}$,
- vertical labels $\in \{-1, 0\}$.



The four tiles can be also interpreted as edges of a **finite state transducer** whose states are the vertical colors and input/output symbols of transitions are the top and the bottom colors:



A tiling of an infinite horizontal strip is a **bi-infinite path** whose input symbols and output symbols read the top and bottom colors of the strip. We have enough transitions to allow the transducer to convert $B(\alpha)$ into $B(2\alpha)$.

An analogous construction can be done for **any rational multiplier q**. We can construct the following tiles for all $k \in \mathbb{Z}$ and all α in the domain interval:

$$B_k(\alpha)$$

$$q\lfloor (k-1)\alpha \rfloor - \lfloor q(k-1)\alpha \rfloor \qquad \boxed{} \qquad q\lfloor k\alpha \rfloor - \lfloor qk\alpha \rfloor$$

$$B_k(q\alpha)$$

The tiles multiply by q, and they admit a tiling of a horizontal strip whose top and bottom labels read $B(\alpha)$ and $B(q\alpha)$.

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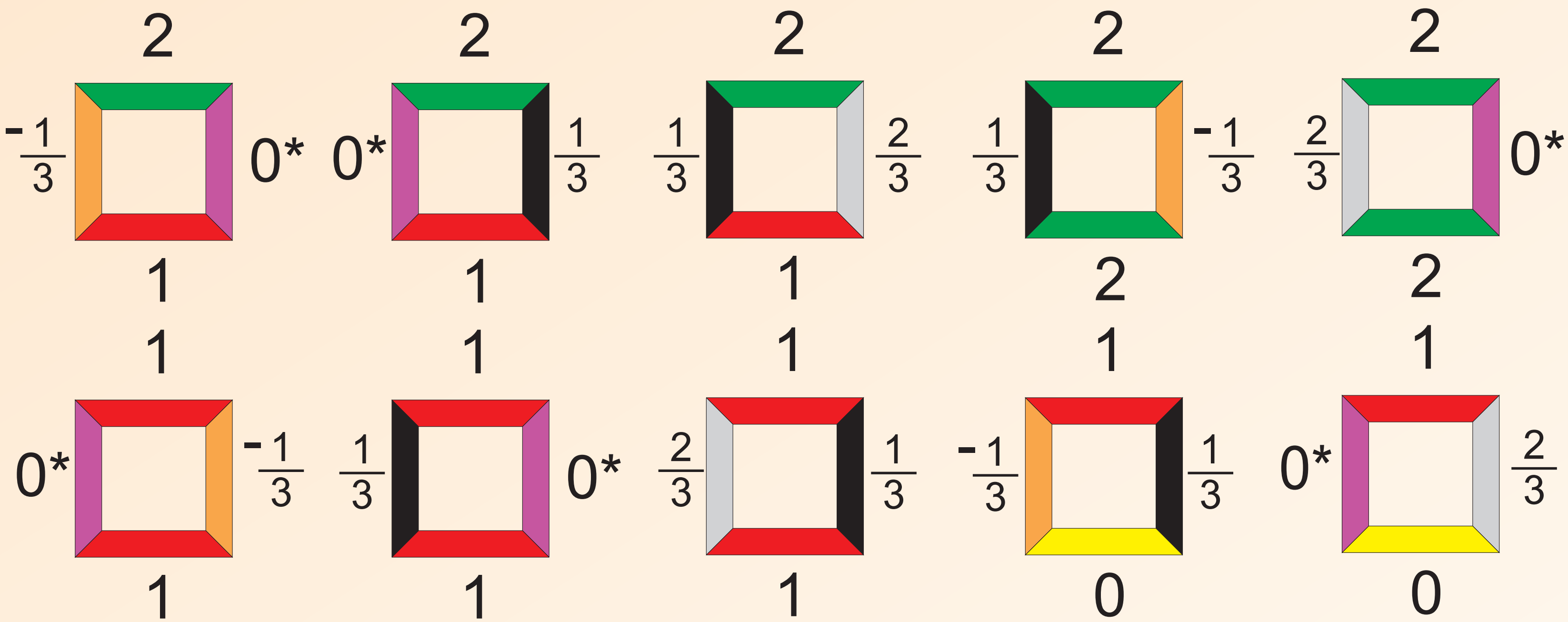
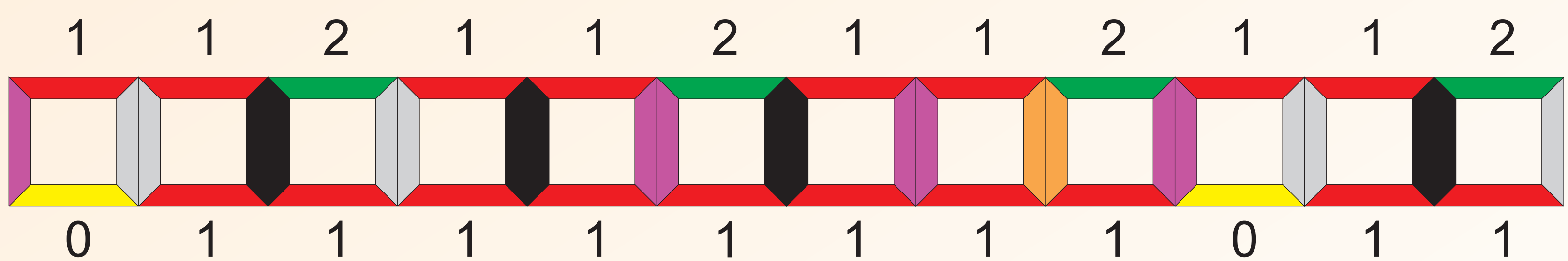
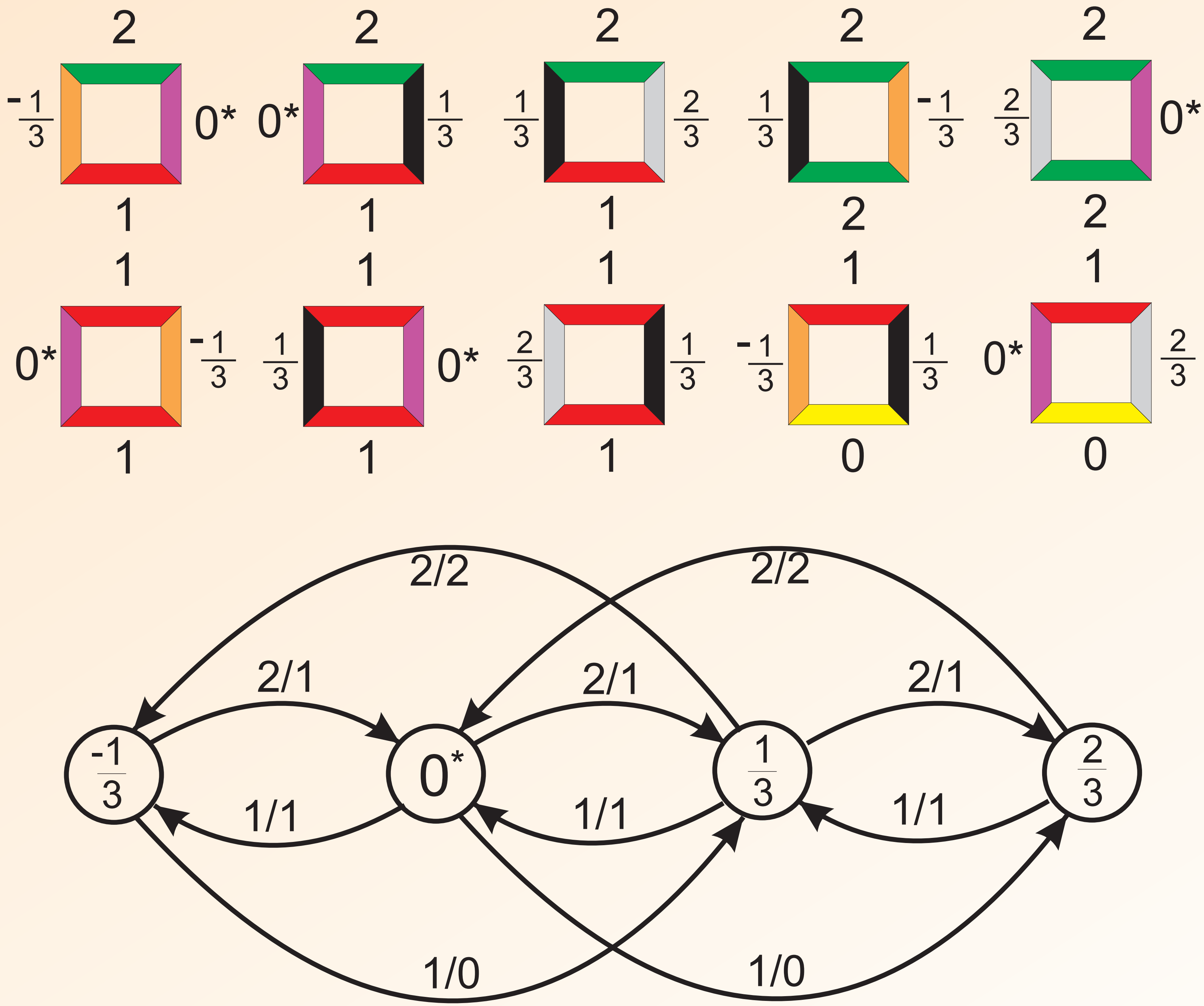
The tiles multiply by q, and they admit a tiling of a horizontal strip whose top and bottom labels read $B(\alpha)$ and $B(q\alpha)$.

Our second tile set $T_{2/3}$ was constructed in this way for $q = \frac{2}{3}$ and $1 \leq \alpha \leq 2$.



So for all $1 \leq \alpha \leq 2$ one can tile a horizontal row whose top and bottom labels read $B(\alpha)$ and $B(\frac{2}{3}\alpha)$:





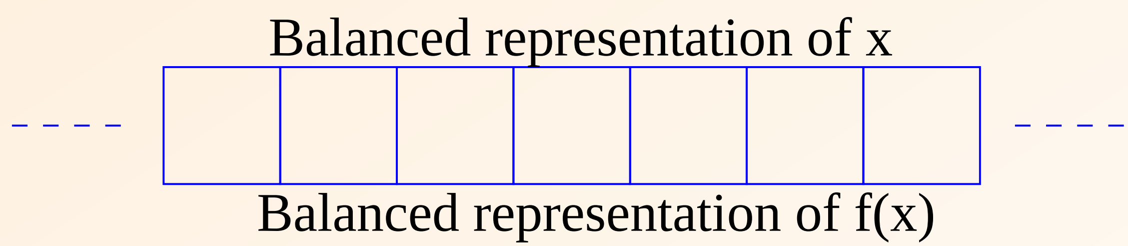
2/2
2/2
2/1
2/1
2/1
1/1
1/1
1/1
1/0
1/0

The tiles admit valid tilings of the plane that simulate iterations of the piecewise linear dynamical system

$$f : [\frac{1}{2}, 2] \longrightarrow [\frac{1}{2}, 2]$$

where

$$f(x) = \begin{cases} 2x, & \text{if } x \in [\frac{1}{2}, 1], \text{ and} \\ \frac{2}{3}x, & \text{if } x \in (1, 2]. \end{cases}$$



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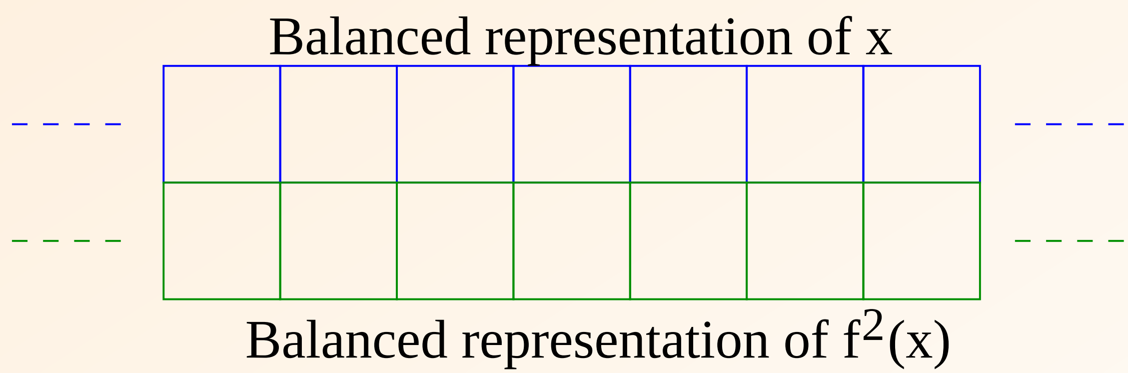

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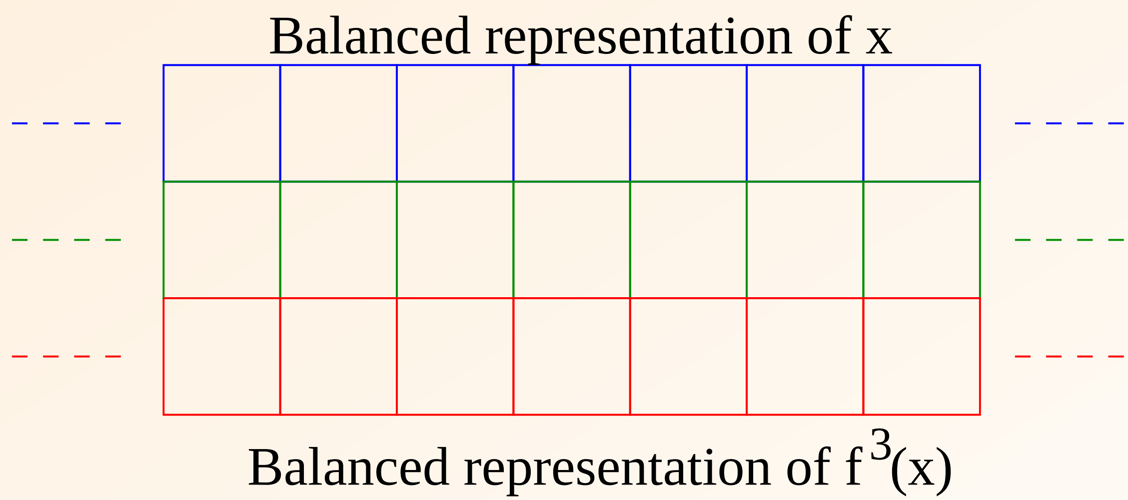

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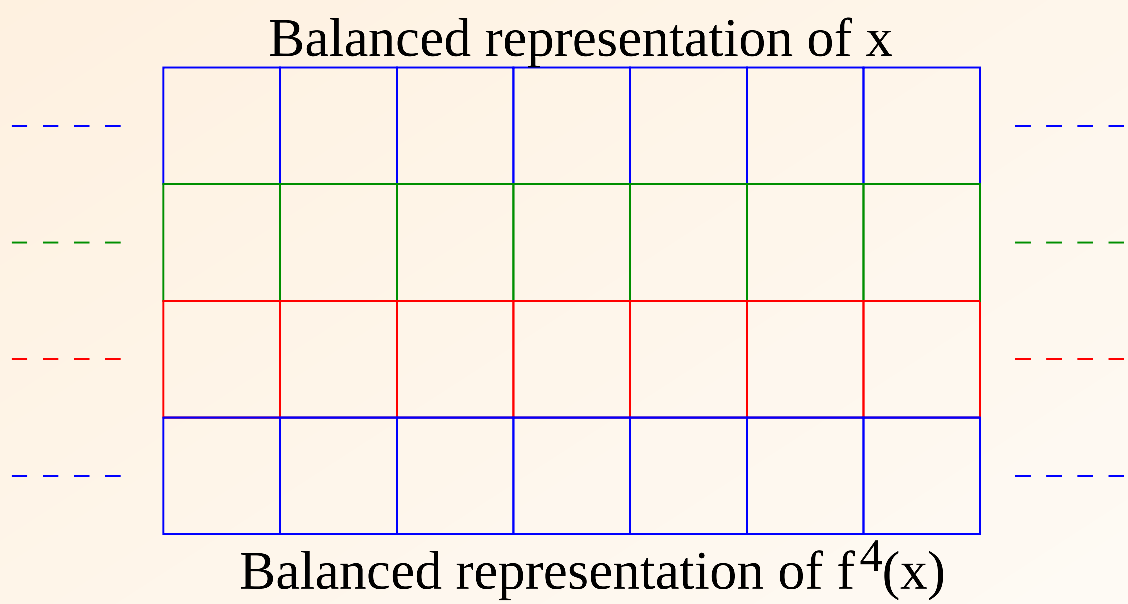

We proved that the 14 tiles are aperiodic.

The construction can be effectively carried out for any piecewise linear function on a union of finite intervals of $\mathbb{R}$, as long as the multiplications are with rational numbers q.

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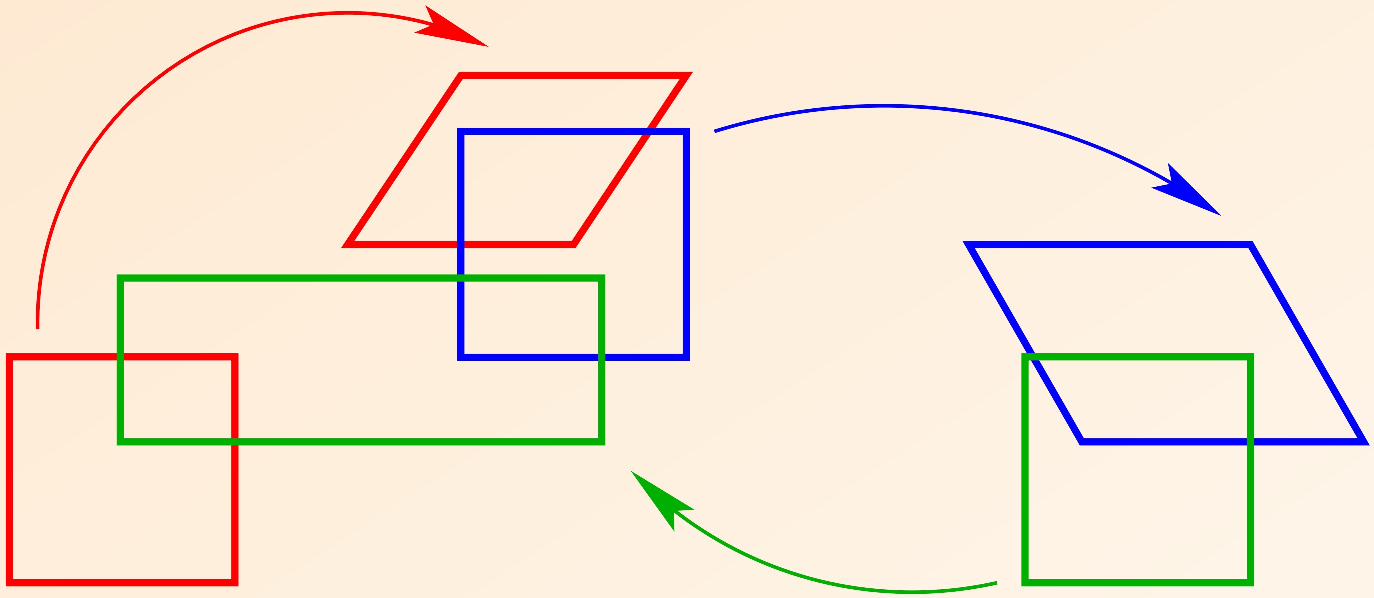
In order to prove **undecidability results** we want to simulate more complex dynamical systems that can carry out Turing computations.

One can **generalize the construction** in two ways:

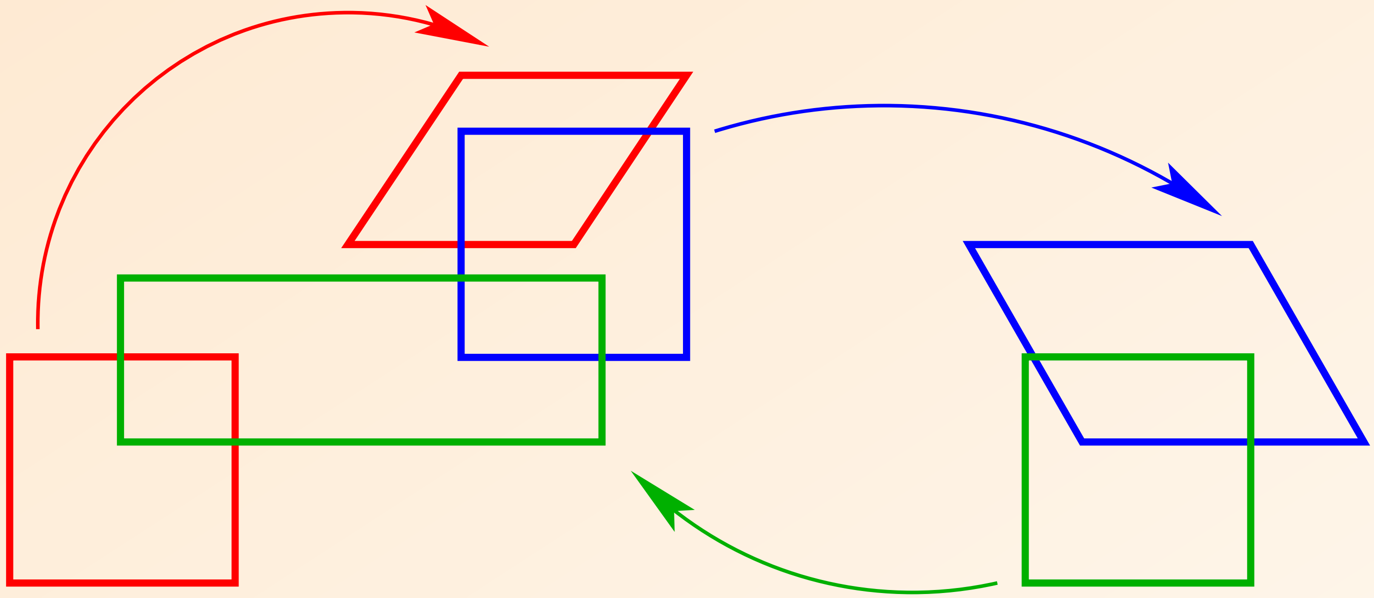
- from linear maps to affine maps, and
- from $\mathbb{R}$ to $\mathbb{R}^2$, (or $\mathbb{R}^d$ for any d).



Consider a system of finitely many pairs (U_i, f_i) where

- U_i are disjoint unit squares of the plane with integer corners,
- f_i are affine transformations with rational coefficients.

Square U_i serves as the domain where f_i may be applied.



The system determines a function

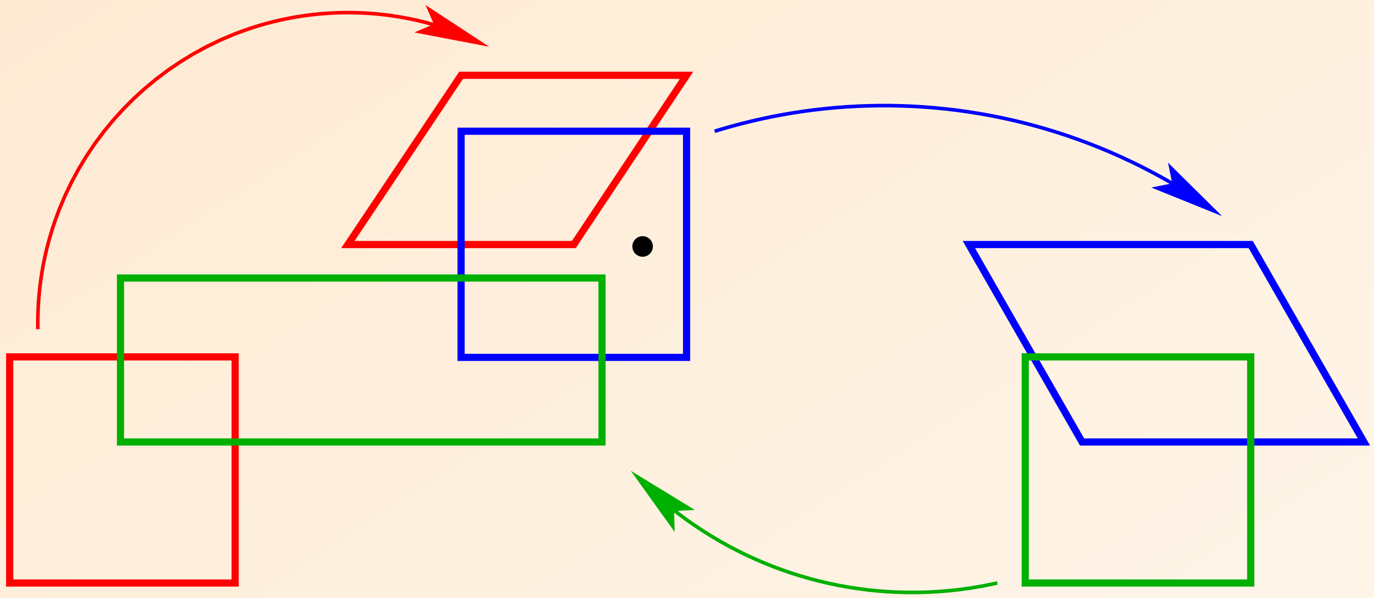
$$f : D \longrightarrow \mathbb{R}^2$$

whose domain is

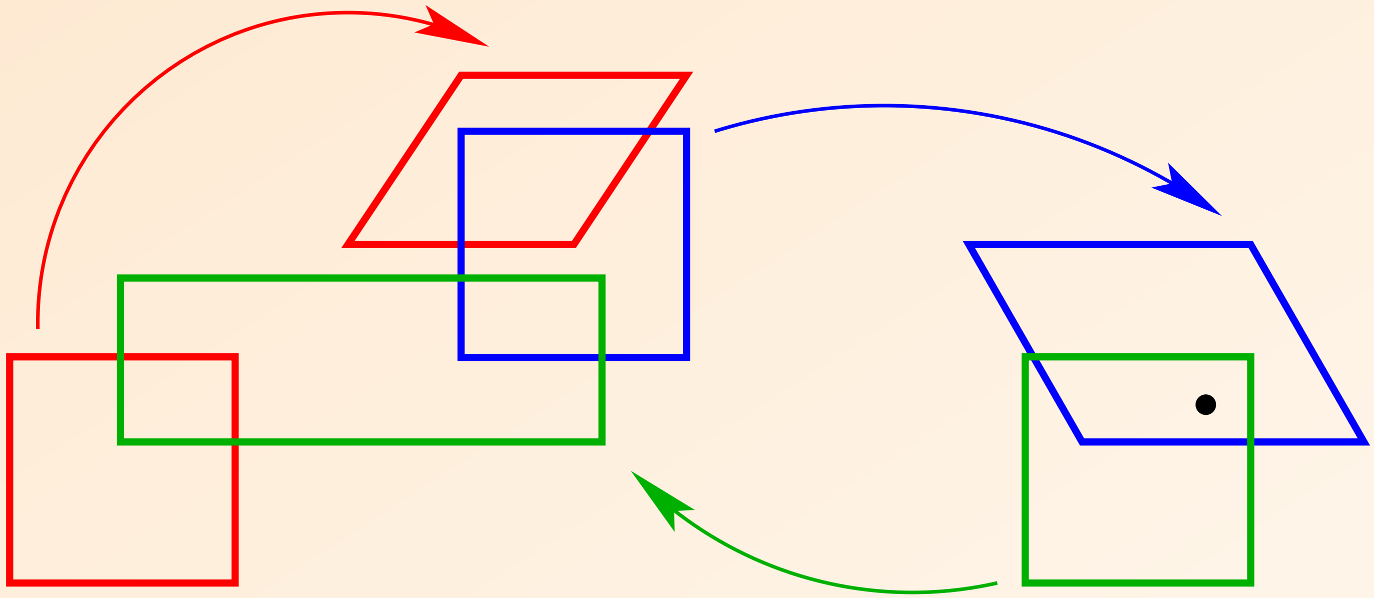
$$D = \bigcup_i U_i$$

and

$$f(\vec{x}) = f_i(\vec{x}) \text{ for all } \vec{x} \in U_i.$$



The orbit of $\vec{x} \in D$ is the iteration of f starting at point $\vec{x}$. The iteration can be continued as long as the point remains in the domain D.



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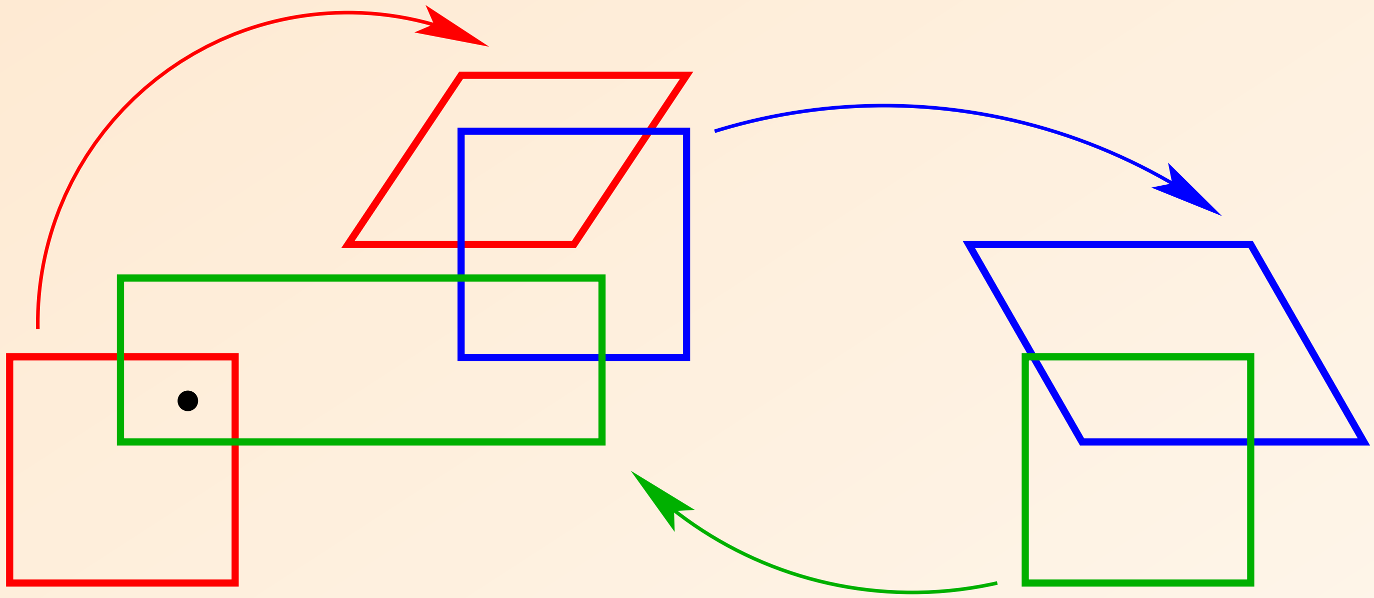

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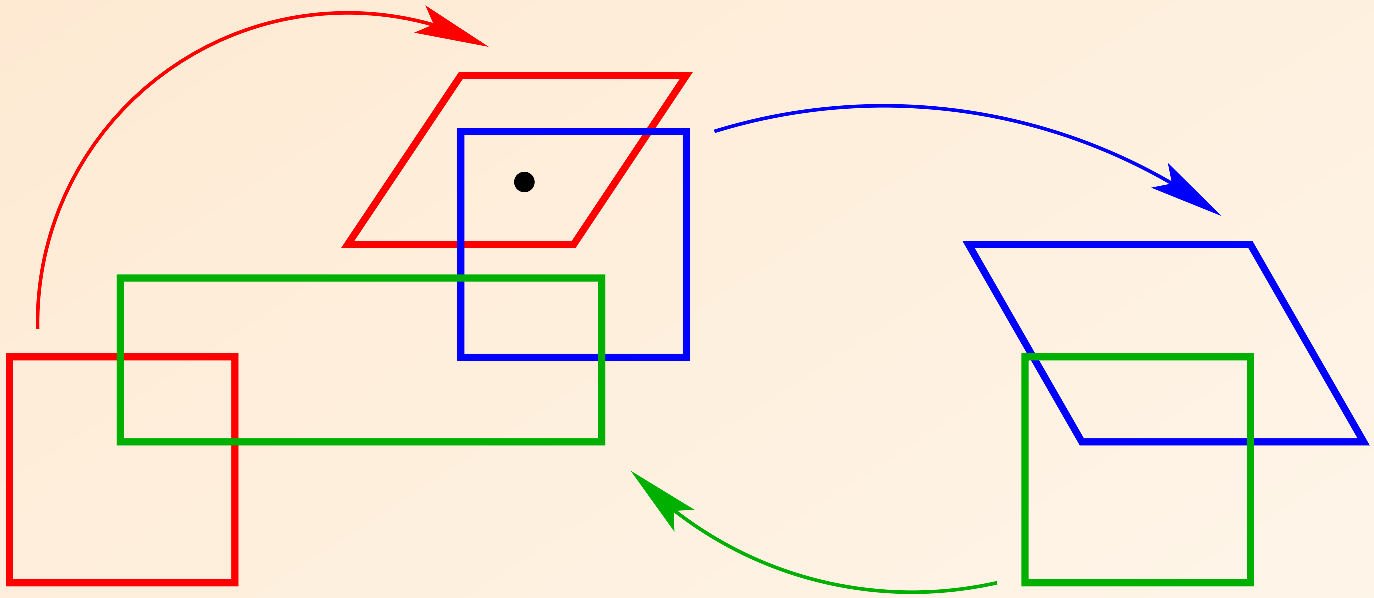

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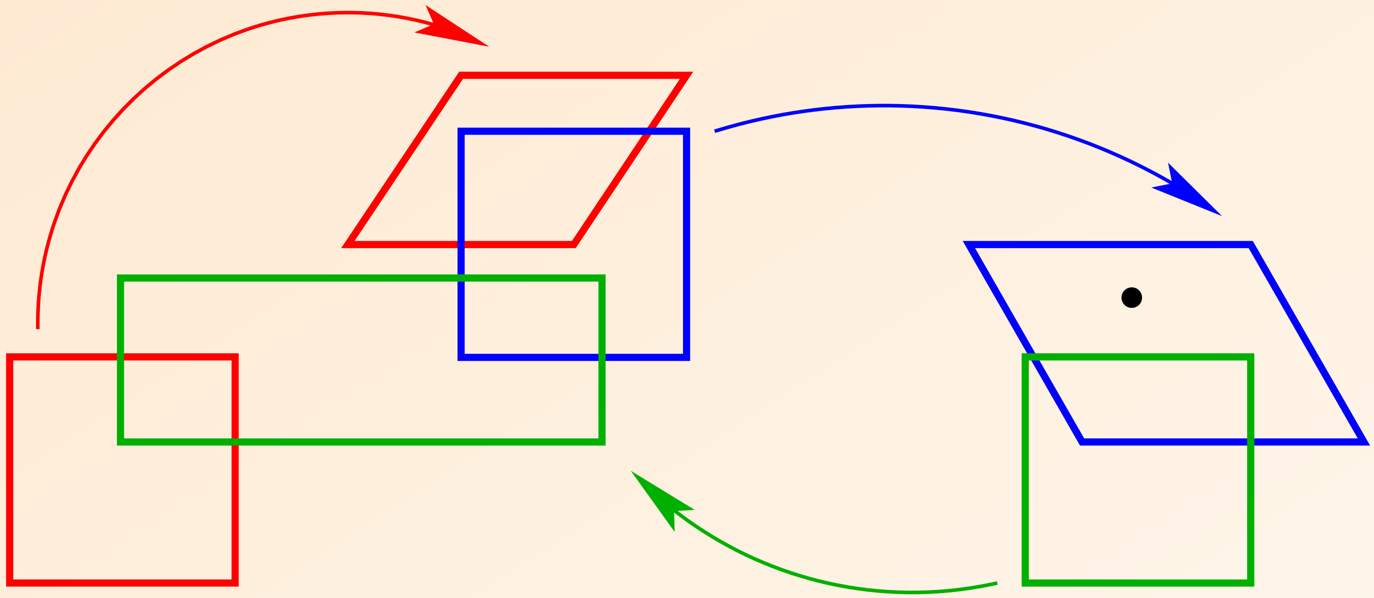
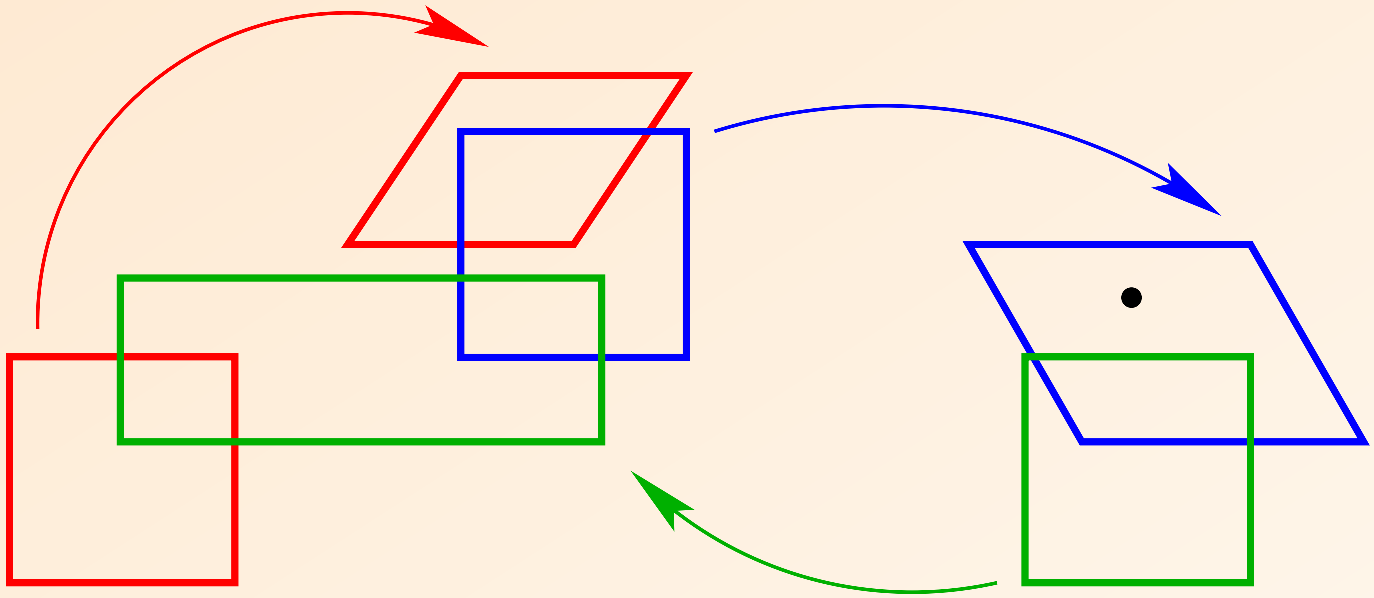


But if the point goes outside of the domain, the system **halts**.

If the iteration always halts, regardless of the starting point $\vec{x}$, the system is **mortal**. Otherwise it is **immortal**: there is an immortal point $\vec{x} \in D$ from which a non-halting orbit begins.



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Immortality problem: Is a given system of affine maps (with rational coordinates) immortal?

Theorem: The immortality problem is undecidable.

To prove the undecidability one can use a standard technique for transforming **Turing machines** into two-dimensional piecewise affine transformations.

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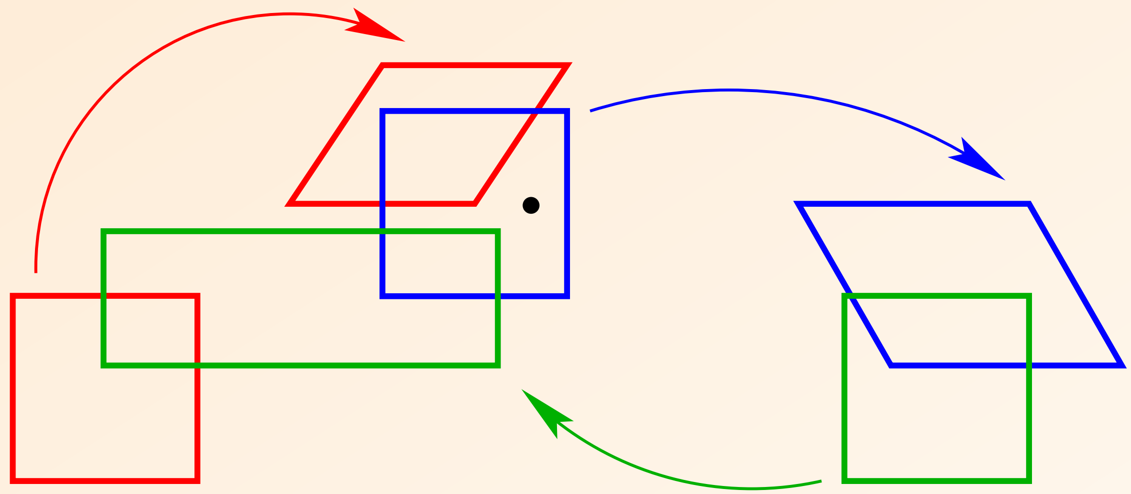
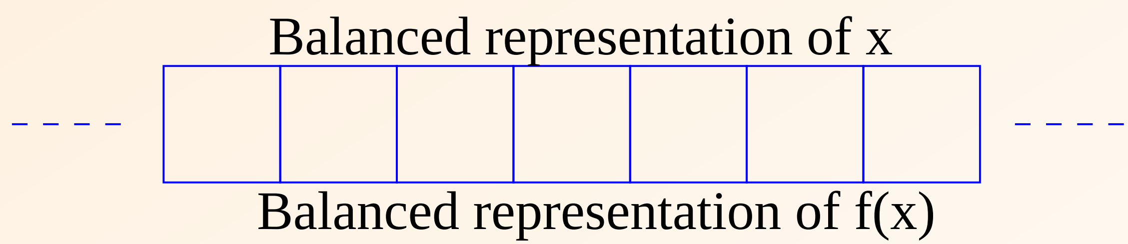
The obtained system of affine maps has an immortal point if and only if the Turing machine has an **immortal configuration,** that is, a (possibly infinite) configuration that leads to a non-halting computation in the Turing machine. But we have the following result:

Theorem (Hooper 1966): It is undecidable if a given Turing machine has any immortal configurations.

Interesting **historical note**: **Hooper** and **Berger** were both students of **Hao Wang** at the same time. Their results are of the same flavor but the proofs are independent.

For any given (rational) system of affine maps, it is possible to effectively construct a set of Wang tiles so that valid tilings necessarily simulate the system:





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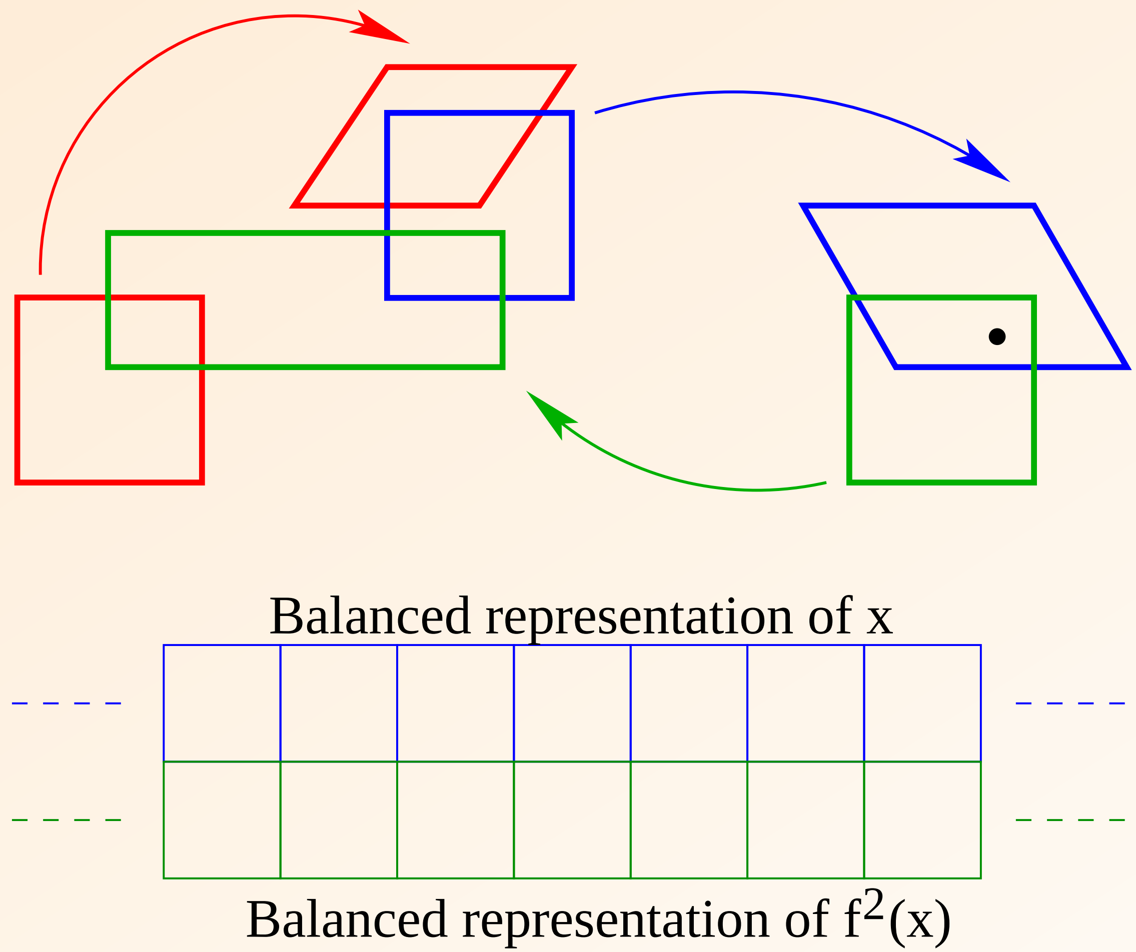

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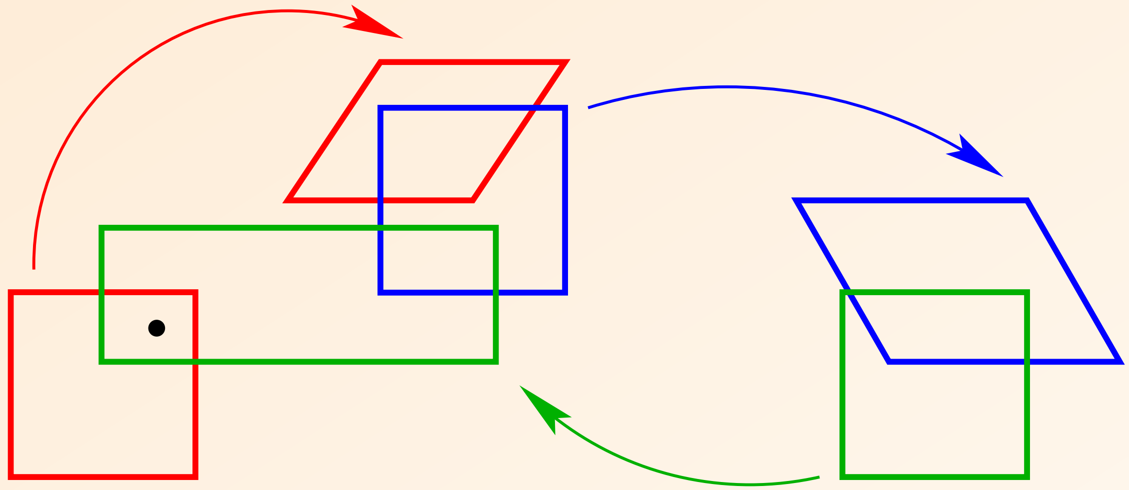
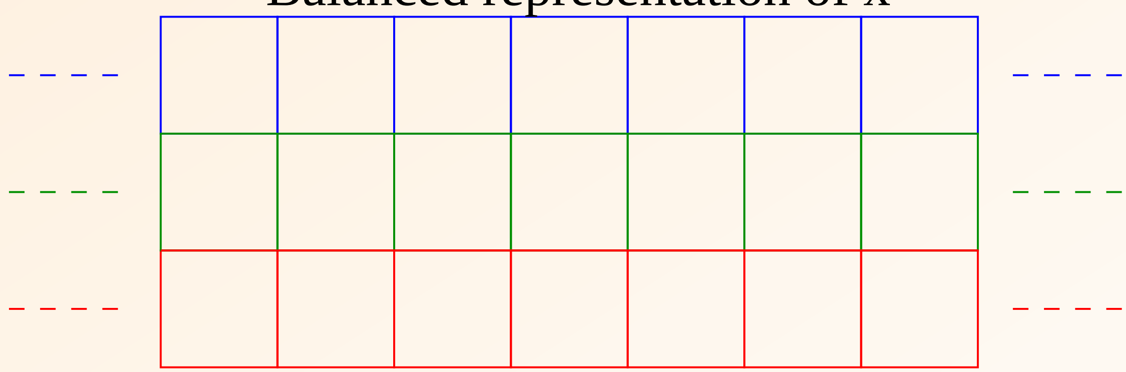


Balanced representation of x



Balanced representation of $f^3(x)$

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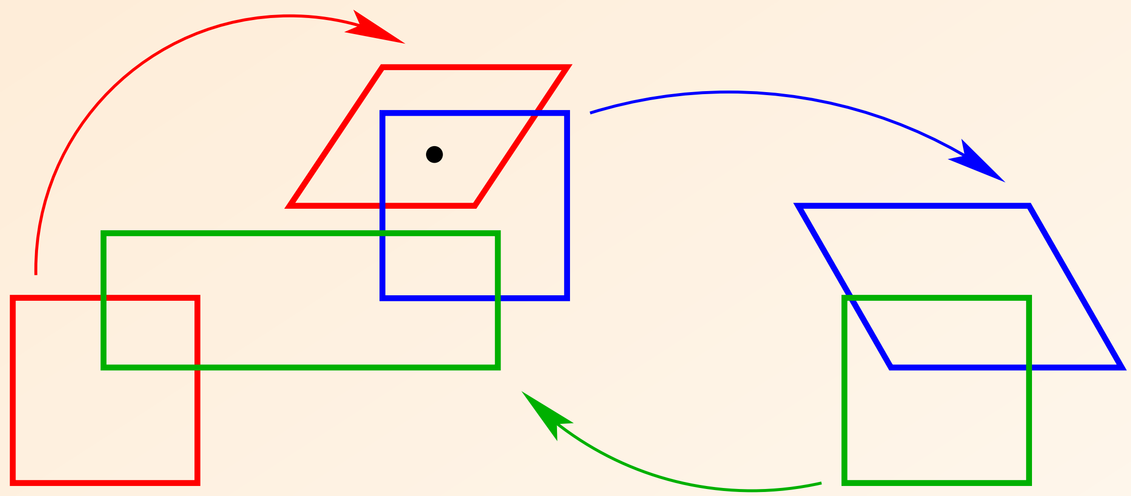

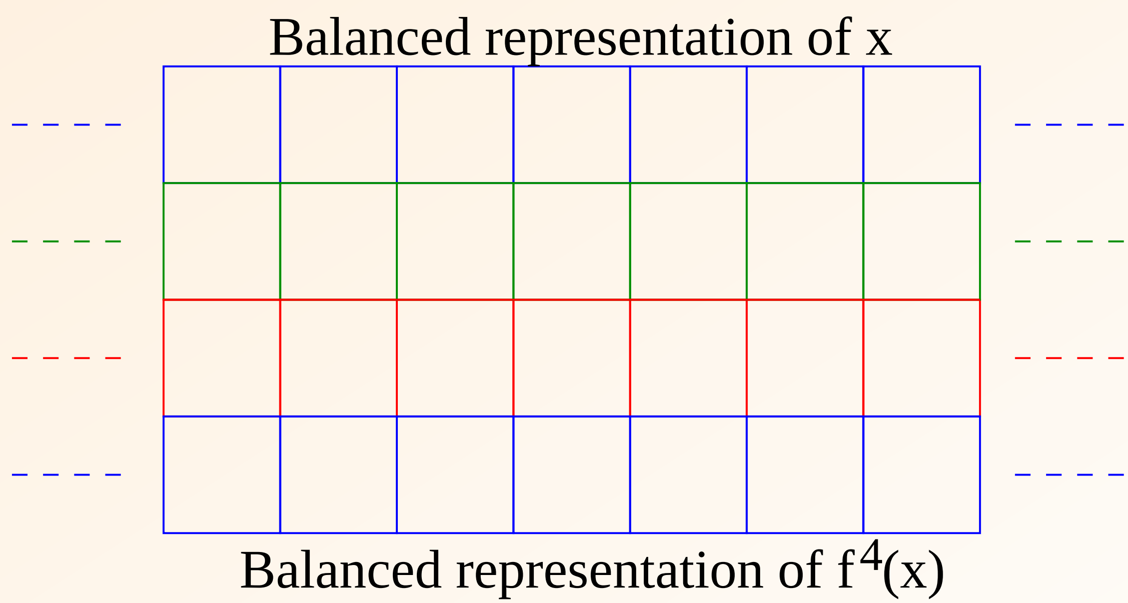

Conclusion: the tile set we constructed admits a tiling of the plane if and only if the system of affine maps is immortal. Undecidability of the tiling problem follows from the undecidability of the immortality problem.

✓