Lecture 4: Effect Handlers

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Outline

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Simple exception handler

- The construct is:

$$M^{\sigma} \text{ handled with } \{\text{raise}_e \mapsto N_e^{\sigma}\}_{e \in E} : \sigma$$

  assuming a finite set of exceptions $E =_{\text{def}} \mathcal{M}[\![\text{exc}]\!]$
- This evidently does not arise from a definable $1 + |E|$-ary operation using the exceptions theory.
- Even worse, it cannot be an operation of any algebraic theory.
- For suppose we have a suitable operation $\text{handle} : \varepsilon; 1, \text{exc}$ say. Then, in general, we will not have:

$$\mathcal{E}[\text{handle}(M, x : \text{exc}.N(x))] = \mathit{handle}(\mathcal{E}[M], x : \text{exc}.\mathcal{E}[N(x)])$$

Failure to be algebraic

Take $\mathcal{E} = (\lambda y : \text{nat}.\, \text{raise}_{e_1})[\,\cdot\,]$, $M = 3$, and $N(x) = \text{raise}_{e_2}$, where $e_1 \neq e_2$. Then we have:

$$\begin{aligned} \models \mathcal{E}[\text{handle}(M, x : \text{exc}.N(x))] &=_{\text{def}} (\lambda y : \text{nat}.\, \text{raise}_{e_1})\text{handle}(3, x : \text{exc}.N(x)) \\ &= (\lambda y : \text{nat}.\, \text{raise}_{e_1})3 \\ &= \text{raise}_{e_1} \end{aligned}$$

and:

$$\begin{aligned} \models \text{handle}(\mathcal{E}[M], x : \text{exc}.\mathcal{E}[N(x)]) & \\ = \text{handle}((\lambda y : \text{nat}.\, \text{raise}_{e_1})3, x : \text{exc}.(\lambda y : \text{nat}.\, \text{raise}_{e_1})N(x)) & \\ = \text{handle}(\text{raise}_{e_1}, x : \text{exc}.(\lambda y : \text{nat}.\, \text{raise}_{e_1})N(x)) & \\ = (\lambda y : \text{nat}.\, \text{raise}_{e_1})N(e_1) & \\ =_{\text{def}} (\lambda y : \text{nat}.\, \text{raise}_{e_1})\text{raise}_{e_2} & \\ = \text{raise}_{e_2} & \end{aligned}$$

and the two are different.

Understanding the Benton and Kennedy exception handler algebraically

Simple exception handler

$$M^{\sigma+E} \text{ handled with } \{\text{raise}_e \mapsto N_e^{\sigma+E}\}_{e\in E} : \sigma + E$$

with E finite. (We are mixing syntax and semantics.)

Benton and Kennedy exception handler

$$M^{\sigma+E} \text{ handled with } \{\text{raise}_e \mapsto N_e^{A}\}_{e\in E} \text{ to } x : \sigma \text{ in } N(x) : A$$

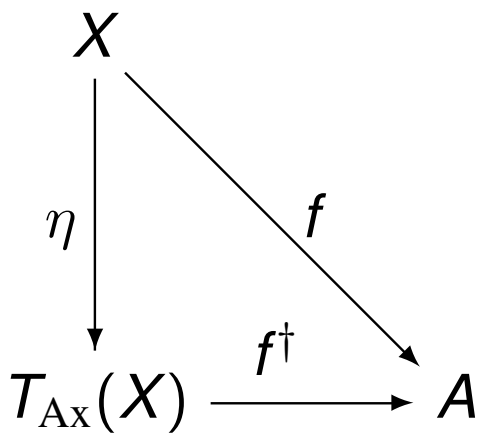
Analysis of the semantics of the BK exception handler

- $M \in T_{\text{Ax}}(\sigma) = \sigma + E$.
- $\{\text{raise}_e \mapsto N_e^A\}_{e\in E}$ specifies a model of Ax with carrier A (any algebra is!).
- $\sigma \xrightarrow{x:\sigma.\, N(x)} A$
- The semantics of the BK exception handler is (that of)

$$(x : \sigma.\, N(x))^\dagger(M)$$

The general algebraic situation

The free model principle, that, for any algebra $\mathcal{A}$ over A satisfying equational axioms Ax, and for any $f : X \to \mathcal{A}$ there exists a unique homomorphism $f^\dagger : T_{\mathrm{Ax}}(X) \to \mathcal{A}$ such that the following diagram commutes



suggests a syntax for, and an interpretation of, *effect deconstructors*. Continuing to mix syntax and semantics, we write:

$$M^{T_{\mathrm{Ax}}(X)} \text{ handled with } \mathcal{A} \text{ to } x : X \text{ in } N(x) : A$$

Additions to the λ-calculus: Handlers for simple operations

- Handlers

$$H ::= \{\mathrm{op}(k_1 : T\tau, \ldots, k_n : T\tau) = B_{\mathrm{op}}\}_{\mathrm{op}:n}$$

where $T(\tau) =_{\mathrm{def}} \mathrm{unit} \to \tau$

$$\frac{\Gamma, k_1 : T\tau, \ldots, k_n : T\tau \vdash B_{\mathrm{op}} : \tau \quad (i = 1, n)}{\Gamma \vdash \{\mathrm{op}(k_1 : T\tau, \ldots, k_n : T\tau) = B_{\mathrm{op}}\}_{\mathrm{op}:n} : \tau \textbf{ handler}}$$

- Handling

$$M ::= M \text{ handled with } H \text{ to } x : \sigma \text{ in } N$$

$$\frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash H : \tau \textbf{ handler} \quad \Gamma, x : \sigma \vdash N : \tau}{\Gamma \vdash M \text{ handled with } H \text{ to } x : \sigma \text{ in } N : \tau}$$

Warning!! Not all handlers are *correct*, i.e., define models, so semantics of a λ-calculus term may not be defined.

How the handlers work

Suppose

$$H ::= \{\mathrm{op}(k_1 : T\tau, \ldots, k_n : T\tau) = B_{\mathrm{op}}(k_1, \ldots, k_n)\}_{\mathrm{op}:n}$$

then

$$x : \sigma \vdash x \text{ handled with } H \text{ to } x : \sigma \text{ in } N = N$$

and

$$\vdash \mathrm{op}(M_1, \ldots, M_n) \text{ handled with } H \text{ to } x : \sigma \text{ in } N = B_{\mathrm{op}}(K_1, \ldots, K_n)$$

where

$$K_i = [M_i \text{ handled with } H \text{ to } x : \sigma \text{ in } N]$$

(and $[M] = \lambda x : \mathrm{unit}.\, M$)

An example: changing the contents of a read-only memory, holding a boolean

Assume there is only one location, storing booleans.

- The handler A " temporary state" handler H_{ro} is given by:

$$b : \mathrm{bool} \vdash \{\mathrm{lookup}(k_1 : T(\tau), k_2 : T(\tau)) = \mathrm{if}\ b\ \mathrm{then}\ k_1(*)\ \mathrm{else}\ k_2(*)\} : \tau\ \textbf{handler}$$

- Handling To evaluate a computation $\vdash M : \sigma$, continuing with $x : \sigma \vdash N : \tau$ and forcing any lookup's to give a value b we use:

$$b : \mathrm{bool} \vdash M\ \mathrm{handled\ with}\ H_{\mathrm{ro}}\ \mathrm{to}\ x : \sigma\ \mathrm{in}\ N : \tau$$

One may prefer a syntax allowing parametric handlers parameterised on arbitrary types.

How this handler works

The handler H_{ro} is:

$$\{\mathrm{lookup}(k_1 : T(\tau), k_2 : T(\tau)) \quad = \quad \text{if } b \text{ then } k_1(*) \text{ else } k_2(*) \ \}$$

It works as follows:

$$\vdash \mathrm{lookup}(M_1, M_2) \text{ handled with } H_{\mathrm{ro}} \text{ to } x : \sigma \text{ in } N$$

$$= \text{if } b \text{ then } [M_1 \text{ handled with } H_{\mathrm{ro}} \text{ to } x : \sigma \text{ in } N](*)$$
$$\text{else } [M_2 \text{ handled with } H_{\mathrm{ro}} \text{ to } x : \sigma \text{ in } N](*)$$

$$= \text{if } b \text{ then } M_1 \text{ handled with } H_{\mathrm{ro}} \text{ to } x : \sigma \text{ in } N$$
$$\text{else } M_2 \text{ handled with } H_{\mathrm{ro}} \text{ to } x : \sigma \text{ in } N$$

Additions to the λ-calculus: General operations

Handlers

$$H ::= \{\mathrm{op}_{\mathbf{x}:\mathbf{s}}(k_1 : \mathbf{s}_1 \to \tau, \ldots, k_n : \mathbf{s}_n \to \tau) = B_{\mathrm{op}}\}_{\mathrm{op}:\mathbf{s};\mathbf{s}_1,\ldots,\mathbf{s}_m}$$

$$\frac{\Gamma, \mathbf{x} : \mathbf{s}, k_1 : \mathbf{s}_1 \to \tau, \ldots, k_n : \mathbf{s}_n \to \tau \vdash B_{\mathrm{op}} : \tau \quad (i = 1, n)}{\Gamma \vdash \{\mathrm{op}_{\mathbf{x}:\mathbf{s}}(k_1 : \mathbf{s}_1 \to \tau, \ldots, k_n : \mathbf{s}_n \to \tau) = B_{\mathrm{op}}\}_{\mathrm{op}:\mathbf{s};\mathbf{s}_1,\ldots,\mathbf{s}_m} : \tau \,\mathbf{handler}}$$

How these handlers work

Suppose H is

$$\{\mathrm{op}_{\mathbf{x}:\mathbf{s}}(k_1 : \mathbf{s}_1 \to \tau, \ldots, k_n : \mathbf{s}_n \to \tau) = B_{\mathrm{op}}(\mathbf{x}, k_1, \ldots, k_n)\}_{\mathrm{op}:\mathbf{s};\mathbf{s}_1,\ldots,\mathbf{s}_m}$$

then

$$\begin{aligned} \vdash \mathrm{op}_{\mathbf{A}}(\mathbf{x}_1 : \mathbf{s}_1.\, M_1, \ldots, \mathbf{x}_n : \mathbf{s}_n.M_n) &\text{ handled with } H \text{ to } x : \sigma \text{ in } N \\ = \text{let } \mathbf{x} : \mathbf{s} &\text{ be } \mathbf{A} \text{ in } B_{\mathrm{op}}(\mathbf{x}, K_1, \ldots, K_n) \end{aligned}$$

where

$$K_i = \lambda \mathbf{x} : \mathbf{s}.\, M_i \text{ handled with } H \text{ to } x : \sigma \text{ in } N$$

An example: rollback

When a computation raises an exception while modifying the memory, e.g., when a connection drops during a database transaction, we may want to revert all modifications made during the computation. This behaviour is termed rollback.

- Signature The (disjoint) union of that for (global) state and exceptions.
- Axioms The union of the two sets of equations for global state and for exceptions, together with two commutation equations:

$$\mathrm{lookup}_l(m : \mathrm{nat}.\,\mathrm{raise}_e) = \mathrm{raise}_e$$

$$\mathrm{update}_{l,v}(\mathrm{raise}_e) = \mathrm{raise}_e$$

  of which the first is redundant.
- Monad

$$T(X) = ((S \times X) + E)^S$$

Exception handler for rollback

Assume there is only one location l_0 .

- The handler A “rollback to n" handler H_{rollback} is given by:

$$n : \text{nat} \vdash \text{raise}_{e:\text{exc}} = \text{update}_{l_0,n}(N_{\text{raise}}(e)) : \tau \textbf{ handler}$$

  where $e : \text{exc} \vdash N_{\text{raise}} : \tau$.

- Handling To evaluate a computation $\vdash M : \sigma$, continuing with $x : \sigma \vdash N : \tau$ if no exception is raised, and otherwise rolling back to the initial state and executing N_{raise} with the exception raised, we use:

$$\vdash \text{lookup}_{l_0}(n : \text{nat}.\, M \text{ handled with } H_{\text{rollback}} \text{ to } x : \sigma \text{ in } N)) : \tau$$

Note: One again may prefer a syntax allowing parametric handlers parameterised on arbitrary types.
In the above one would then take n as a parameter, rather than a free variable, and replace N_{raise} by a parameter of type $\text{exc} \to \tau$.

Additions to the λ-calculus: Handlers with parameters for simple operations

- Handlers

$$H ::= \{\mathrm{op}(k_1 : \pi \to \tau, \ldots, k_n, : \pi \to \tau)\,@\,p : \pi = B_{\mathrm{op}}\}_{\mathrm{op}:n}$$

$$\frac{\Gamma, k_1 : \pi \to \tau, \ldots, k_n : \pi \to \tau, p : \pi \vdash B_{\mathrm{op}} : \tau \quad (\mathrm{op} : n)}{\Gamma \vdash \{\mathrm{op}(k_1 : \pi \to \tau, \ldots, k_n : \pi \to \tau)\,@\,p : \pi = B_{\mathrm{op}}\}_{\mathrm{op}:n} : \pi \to \tau\ \textbf{handler}}$$

- Handling

$$M ::= M \text{ handled with } H@P \text{ to } x : \sigma \text{ in } N$$

$$\frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash H : \pi \to \tau\ \textbf{handler} \quad \Gamma \vdash P : \pi \quad \Gamma, x : \sigma \vdash N : \tau}{\Gamma \vdash M \text{ handled with } H@P \text{ to } x : \sigma \text{ in } N : \tau}$$

How parameterised simple handlers work

Suppose H is

$$\{\mathrm{op}(k_1 : \pi \to \tau, \ldots, k_n, : \pi \to \tau)\,@\,p : \pi = B_{\mathrm{op}}(p, k_1, \ldots, k_n)\}_{\mathrm{op}:n}$$

then

$$\begin{aligned} \vdash \mathrm{op}(M_1, \ldots, M_n) \text{ handled with } H@P \text{ to } x : \sigma \text{ in } N \\ = \text{let } p : \pi \text{ be } P \text{ in } B_{\mathrm{op}}(p, K_1, \ldots, K_n) \end{aligned}$$

where

$$K_i = \lambda p : \pi.\, M_i \text{ handled with } H@p \text{ to } x : \sigma \text{ in } N$$

A parameterised handler example: changing the contents of a boolean read-only memory

Assume there is only one location, storing booleans.

- The handler The handler H_{ro} is (now):

$$\{\mathrm{lookup}(k_1 : \mathrm{bool} \to \tau, k_2 : \mathrm{bool} \to \tau) \mathbin{@} b : \mathrm{bool} \\ = \mathrm{if}\ b\ \mathrm{then}\ k_1(b)\ \mathrm{else}\ k_2(b)\} : \mathrm{bool} \to \tau\ \textbf{handler}$$

- Handling To evaluate a computation $\vdash M : \sigma$, continuing with $x : \sigma \vdash N : \tau$ and forcing any lookup's to give a value P we use:

$$\vdash\ M\ \mathrm{handled\ with}\ H_{\mathrm{ro}} @ P\ \mathrm{to}\ x : \sigma\ \mathrm{in}\ N : \tau$$

How this handler works

The handler H_{ro} is:

$$\{\mathrm{lookup}(k_1 : \mathrm{bool} \to \tau, k_2 : \mathrm{bool} \to \tau)\,@\,b : \mathrm{bool} \\ = \mathrm{if}\ b\ \mathrm{then}\ k_1(b)\ \mathrm{else}\ k_2(b)\}$$

It works as follows:

$$\vdash \mathrm{lookup}(M_1, M_2)\ \mathrm{handled\ with}\ H_{\mathrm{ro}}@\mathrm{true}\ \mathrm{to}\ x : \sigma\ \mathrm{in}\ N$$

$$= \mathrm{if\ true\ then}\ (\lambda b : \mathrm{bool}.\ M_1\ \mathrm{handled\ with}\ H_{\mathrm{ro}}@b\ \mathrm{to}\ x : \sigma\ \mathrm{in}\ N)(\mathrm{true}) \\ \mathrm{else}\ (\lambda b : \mathrm{bool}.\ M_2\ \mathrm{handled\ with}\ H_{\mathrm{ro}}@b\ \mathrm{to}\ x : \sigma\ \mathrm{in}\ N)(\mathrm{true})$$

$$= M_1\ \mathrm{handled\ with}\ H_{\mathrm{ro}}@\mathrm{true}\ \mathrm{to}\ x : \sigma\ \mathrm{in}\ N$$

Faking output and curtailing input

The handler H is

$$\begin{aligned}
&\{\mathrm{input}(k : \mathrm{nat} \to (\mathrm{in} \to \tau))@\mathrm{limit} : \mathrm{nat} \\
&\quad = \mathrm{if}\ \mathrm{limit} > 0\ \mathrm{then}\ \mathrm{input}(y : \mathrm{in}.\ k(\mathrm{limit} - 1)(y)) \\
&\qquad\qquad \mathrm{else}\ \mathrm{raise}_{\mathrm{input_session_finished}}(), \\
&\quad \mathrm{output}_{z:\mathrm{out}}(k : \mathrm{nat} \to \tau)@\mathrm{limit} : \mathrm{nat} \\
&\quad = \mathrm{output}_{\mathrm{fake}}(k(\mathrm{limit}))\}
\end{aligned}$$

It works as follows:

$$\begin{aligned}
&\vdash \mathrm{input}(y : \mathrm{in}.\ M)\ \mathrm{handled\ with}\ H@\mathrm{limit}\ \mathrm{to}\ x : \sigma\ \mathrm{in}\ N \\
&\quad = \mathrm{if}\ \mathrm{limit} > 0 \\
&\qquad \mathrm{then}\ \mathrm{input}(y : \mathrm{in}.\ M\ \mathrm{handled\ with}\ H@(\mathrm{limit}-1)\ \mathrm{to}\ x : \sigma\ \mathrm{in}\ N) \\
&\qquad \mathrm{else}\ \mathrm{raise}_{\mathrm{input_session_finished}}()
\end{aligned}$$

$$\begin{aligned}
&\vdash \mathrm{output}_3(M)\ \mathrm{handled\ with}\ H@\mathrm{limit}\ \mathrm{to}\ x : \sigma\ \mathrm{in}\ N \\
&\quad = \mathrm{output}_{\mathrm{fake}}(M\ \mathrm{handled\ with}\ H@\mathrm{limit}\ \mathrm{to}\ x : \sigma\ \mathrm{in}\ N)
\end{aligned}$$

A possible treatment of handlers for effects and types

- Effects

$$\alpha \subseteq_{\text{fin}} \Sigma$$

- Handling

$$M ::= M \text{ handled with } H \text{ to } x : \sigma \text{ in } N$$

$$\frac{\Gamma \vdash M : \sigma ! \alpha \quad \Gamma \vdash H : \alpha \, to \, \tau ! \beta \, \textbf{handler} \quad \Gamma, x : \sigma \vdash N : \tau ! \beta}{\Gamma \vdash M \text{ handled with } H \text{ to } x : \sigma \text{ in } N : \tau ! \beta}$$

- Handlers

$$H ::= \{\text{op}(x_1 : T_\beta(\tau), \ldots, x_n : T_\beta(\tau)) \mapsto B_{\text{op}}\}_{\text{op}:n \in \alpha}$$

where $T_\beta(\tau) =_{\text{def}} \text{unit} \xrightarrow{\beta} \tau$

$$\frac{\Gamma, x_1 : T_\beta(\tau), \ldots, x_n : T_\beta(\tau) \vdash B_{\text{op}} : \tau ! \beta \quad (i = 1, n)}{\Gamma \vdash \{\text{op}(x_1 : T_\beta(\tau), \ldots, x_n : T_\beta(\tau)) \mapsto B_{\text{op}}\}_{\text{op}:n \in \alpha} : \alpha \, to \, \tau ! \beta \, \textbf{handler}}$$

Discussion

- The above language is maximal in that arbitrary handlers can be defined. These define interpretations, but not necessarily models. It is up to the programmer to not write meaningless programs.
- One might instead add a proof requirement, à la type theory, so that a program is not well-formed unless a proof has been supplied.
- One might instead consider a two-level version in which only the compiler writers write handlers. Plotkin and Pretnar, ESOP.
- One might consider restricting the handlers that can be written, so that only meaningful programs can be written. Buneman et al comprehension syntax for database programming on collections (= bags = elements of free commutative monoids).
- If one works only with free algebras, so not "real" effects, then all programs are correct and one has an operational semantics. Might be handy programming idiom. Bauer and Pretnar *Eff*

Outline

Finite Nondeterminism with deadlock

Working in **Set** we take $T(X) = \mathcal{F}(X)$ the collection of finite subsets of X to model nondeterminism, including an “empty" choice (deadlock).

To create the effects we add two *effect constructors*:

$$\frac{M : \sigma \quad N : \sigma}{M + N : \sigma} \qquad \mathrm{NIL} : \sigma$$

Nondeterminism as an algebraic effect

There is a natural equational theory, with signature $+ : 2 \to 1$, $\mathrm{NIL} : 0 \to 1$ and axioms:

$$\begin{array}{lccc} \textit{Associativity} & (x + y) + z & = & x + (y + z) \\ \textit{Commutativity} & x + y & = & y + x \\ \textit{Absorption} & x + x & = & x \\ \textit{Zero} & \mathit{NIL} + x & = & x \end{array}$$

The evident algebra on $\mathcal{F}(X)$ satisfies these equations, interpreting $+$ as $\cup$, and NIL as $\emptyset$.

Further: $\mathcal{F}$ is the free algebra monad.

CCS

Syntax

$$P ::= a.P\ (a \in \mathrm{Act}) \mid P + Q \mid \mathrm{NIL} \mid P\backslash a \mid P[a/b] \mid P|Q$$

Effect Constructors

Unary Effect Deconstructors handling P

Binary Effect Deconstructors handling P and Q

Equational theory for the constructors

Signature: $a.- : 1 \to 1$, for $a \in \mathrm{Act}$, $+ : 2 \to 1$, $\mathrm{NIL} : 0 \to 1$

Axioms: That $+, \mathrm{NIL}$ forms a commutative semilattice, as per finite nondeterminism with deadlock.

Modelling CCS

We model CCS terms as elements of $\mathrm{ST} =_{\mathrm{def}} T_{\mathrm{CCS}}(\emptyset)$; these are just the finite synchronisation trees.

The Restriction Deconstructor

Restriction

$$-\backslash a : \mathrm{ST} \to \mathrm{ST}$$

is the unique homomorphism

$$-\backslash a : \mathrm{ST} \longrightarrow A$$

where A is the algebra with carrier ST and operations given by:

$$\begin{array}{lcl} (b._A u) & = & \begin{cases} \mathrm{NIL} & (b = a) \\ b.u & (b \neq a) \end{cases} \\ +_A(u, v) & = & u + v \\ \mathrm{NIL}_A & = & \mathrm{NIL} \end{array}$$

Note This evidently defines a CCS-algebra.

The Restriction Deconstructor (cntnd.)

More intuitively, one can simply define restriction by a kind of primitive recursion.

We have:

$$
\begin{array}{lcll}
(b.u)\backslash a & = & b._A(u\backslash a) & = \begin{cases} \mathrm{NIL} & (b = a) \\ b.(u\backslash a) & (b \neq a) \end{cases} \\
(u + v)\backslash a & = & u\backslash a +_A v\backslash a & = u\backslash a + v\backslash a \\
\mathrm{NIL}\backslash a & = & \mathrm{NIL}_A & = \mathrm{NIL}
\end{array}
$$

The Restriction Deconstructor (cntnd.)

So we can just define restriction by:

$$\begin{array}{lcl} (b.u)\backslash a & = & \begin{cases} \mathrm{NIL} & (b = a) \\ b.(u\backslash a) & (b \neq a) \end{cases} \\ (u+v)\backslash a & = & u\backslash a + v\backslash a \\ \mathrm{NIL}\backslash a & = & \mathrm{NIL} \end{array}$$

But one needs also to verify that the implicit algebra on ST is a CCS-Algebra.

Remark: This restriction is not exactly that of CCS. It is an exercise to correct the definition.

The Renaming Deconstructor

This is defined recursively by:

$$
\begin{aligned}
(c.u)[a/b] &= \begin{cases} a.(u[a/b]) & (c = b) \\ c.(u[a/b]) & (c \neq b) \end{cases} \\
(u + v)[a/b] &= u[a/b] + v[a/b] \\
\mathrm{NIL}[a/b] &= \mathrm{NIL}
\end{aligned}
$$

(and, as before, a correction needs to be made to get CCS renaming).

Interleaving

Consider the interleaving function

$$\mid : \mathrm{ST} \times \mathrm{ST} \longrightarrow \mathrm{ST}$$

Following the rather natural Dutch ACP approach, we write it as the sum of left interleaving and right interleaving operations:

$$u \mid v = u \mid^{\mathrm{l}} v + u \mid^{\mathrm{r}} v$$

where $\mid^{\mathrm{l}}$ has first action that of its first argument and then becomes a regular interleaving, and $\mid^{\mathrm{r}}$ rather favours its second argument.

Interleaving Defined

There is a natural "mutually recursive" definition:
The left and right operators satisfy the following defining equations:

$$\begin{array}{rcl} \mathrm{NIL} \,|^{\mathrm{l}}\, z & = & \mathrm{NIL} \\ (x + y) \,|^{\mathrm{l}}\, z & = & (x \,|^{\mathrm{l}}\, z) + (y \,|^{\mathrm{l}}\, z) \\ a.x \,|^{\mathrm{l}}\, z & = & a.(x \,|^{\mathrm{l}}\, z + x \,|^{\mathrm{r}}\, z) \end{array}$$

and

$$\begin{array}{rcl} z \,|^{\mathrm{r}}\, \mathrm{NIL} & = & \mathrm{NIL} \\ z \,|^{\mathrm{r}}\, (x + y) & = & (z \,|^{\mathrm{r}}\, x) + (z \,|^{\mathrm{r}}\, y) \\ z \,|^{\mathrm{r}}\, a.x & = & a.(z \,|^{\mathrm{l}}\, x + z \,|^{\mathrm{r}}\, x) \end{array}$$

But these equations do not fit with the homomorphic view (even accommodating it to allow parameters and mutually recursive definitions). The problem is the switch from recursion variable to parameter.

A homomorphic solution to the defining equations

Define

$$\bar{l} : \mathrm{ST} \to \mathrm{ST}^{\mathrm{ST}} \qquad \bar{r} : \mathrm{ST} \times \mathrm{ST}^{\mathrm{ST}} \to \mathrm{ST}$$

as follows:

$$\begin{array}{rcl} \bar{l}(\mathrm{NIL}) & = & \lambda z : \mathrm{ST}.\, \mathrm{NIL} \\ \bar{l}(x+y) & = & \lambda z : \mathrm{ST}.\, \bar{l}(x)(z) + \bar{l}(y)(z) \\ \bar{l}(a.x) & = & \lambda z : \mathrm{ST}.\, a.(\bar{l}(x)(z) + \bar{r}(z, \bar{l}(x))) \end{array}$$

and

$$\begin{array}{rcl} \bar{r}(\mathrm{NIL}, f) & = & \mathrm{NIL} \\ \bar{r}(x+y, f) & = & \bar{r}(x, f) + \bar{r}(y, f) \\ \bar{r}(a.x, f) & = & a.(f(x) + \bar{r}(x, f)) \end{array}$$

and then set:

$$x \,|^{\mathrm{l}}\, z = \bar{l}(x)(z) \qquad x \,|^{\mathrm{r}}\, z = \bar{r}(z, \bar{l}(x))$$

Why this is a solution

Left Shuffle

$$a.x \,|^{\mathrm{l}}\, z = \bar{l}(a.x)(z) = a.(\bar{l}(x)(z) + \bar{r}(z, \bar{l}(x))) = a.(x \,|^{\mathrm{l}}\, z + x \,|^{\mathrm{r}}\, z)$$

Right Shuffle

$$x \,|^{\mathrm{r}}\, a.z = \bar{r}(a.z, \bar{l}(x)) = a.(\bar{l}(x)(z) + \bar{r}(z, \bar{l}(x))) = a.(x \,|^{\mathrm{l}}\, z + x \,|^{\mathrm{r}}\, z)$$

(The idea was independently noted by Paul Levy.)

Dendriform dialgebras

- A dendriform dialgebra (Loday, 1993) is a **k**-vector space $\langle A, + \rangle$ equipped with two binary operations, $\lhd$ and $\rhd$ such that, for all $x, y, z \in A$:

$$\begin{aligned} (x \lhd y) \lhd z &= x \lhd (y \bowtie z) \\ (x \rhd y) \lhd z &= x \rhd (y \lhd z) \\ x \rhd (y \rhd z) &= (x \bowtie y) \rhd z \end{aligned}$$

  where

$$x \bowtie y =_{\text{def}} x \lhd y + x \rhd y$$

- It is commutative (Shützenberger) if $x \lhd y = y \rhd x$ always holds.
- Then $\bowtie$ is an associative operation; it is commutative if the dialgebra is.

Concurrency with synchronisation

- Again following the ACP tradition, split $\mid$ into three parts:

$$x \mid y = x \mid^{\mathrm{l}} y + x \mid^{\mathrm{s}} y + x \mid^{\mathrm{r}} y$$

  where the central $\mid^{\mathrm{s}}$ is for synchronisation.
- An NS algebra (Leroux, 2003) is a **k**-vector space equipped with three bilinear operations, $\triangleleft$ (left-linear), $\triangleright$ (right-linear), and $\bullet$ such that

$$\begin{array}{lcl} (x \triangleleft y) \triangleleft z & = & x \triangleleft (y * z) \\ (x \triangleright y) \triangleleft z & = & x \triangleright (y \triangleleft z) \\ x \triangleright (y \triangleright z) & = & (x * y) \triangleright z \\ (x * y) \bullet z + (x \bullet y) \triangleleft z & = & x \triangleright (y \bullet z) + x \bullet (y * z) \end{array}$$

  where $x * y =_{\mathrm{def}} x \triangleleft y + x \bullet y + x \triangleright y$
- It is commutative if $\bullet$ is and $x \triangleleft y = y \triangleright x$ always holds.
- Then $*$ is an associative bilinear operation; it is commutative if the NS-algebra is.

Dendriform trialgebra, 1984

A dendriform trialgebra (Loday and Ronco, 2004) consists of a **k**-vector space with three, binary operations $\lhd, \rhd$, and $\bullet$ s.t.:

$$\begin{aligned}
(x \lhd y) \lhd z &= x \lhd (y * z) \\
(x \rhd y) \lhd z &= x \rhd (y \lhd z) \\
x \rhd (y \rhd z) &= (x * y) \rhd z
\end{aligned}$$

$$\begin{aligned}
x \bullet (y \lhd z) &= (x \bullet y) \lhd z \\
(x \rhd y) \bullet z &= x \rhd (y \bullet z) \\
(x \lhd y) \bullet z &= x \bullet (y \rhd z)
\end{aligned}$$

$$(x \bullet y) \bullet z = x \bullet (y \bullet z)$$

where $* =_{\text{def}} \lhd + \bullet + \rhd$. It is automatically an NS-algebra.
These equations appear already in Bergstra and Klop, 1984

Concurrency definition

- Synchronisation algebra

  $\langle A, \cdot \rangle$ a commutative partial monoid

- CCS Example

  $\langle \mathrm{Act}, \cdot \rangle$ where:

$$a \cdot b = \begin{cases} \tau & (\overline{a} = b \neq \tau) \\ \uparrow & (\text{otherwise}) \end{cases}$$

Note: We use Roman a, etc, rather than Greek α for CCS actions.

Defining concurrency with synchronisation

- Define $|^{\mathrm{l}}$, $|^{\mathrm{r}}$, and $|^{\mathrm{s}}$, together with $|^{\mathrm{sr}} : A \times \mathrm{ST} \times \mathrm{ST} \longrightarrow \mathrm{ST}$
- by:

$$
\begin{array}{rcl}
a.x \,|^{\mathrm{l}}\, z & = & a.(x \,|^{\mathrm{l}}\, z + x \,|^{\mathrm{s}}\, z + x \,|^{\mathrm{r}}\, z), \text{etc} \\
\\
\mathrm{NIL} \,|^{\mathrm{s}}\, z & = & \mathrm{NIL} \\
(x + y) \,|^{\mathrm{s}}\, z & = & x \,|^{\mathrm{s}}\, z \; + \; y \,|^{\mathrm{s}}\, z \\
a.x \,|^{\mathrm{s}}\, z & = & x \,|^{\mathrm{sr}}{}_a z \\
\\
z \,|^{\mathrm{r}}\, a.y & = & a.(z \,|^{\mathrm{l}}\, y + z \,|^{\mathrm{s}}\, y + z \,|^{\mathrm{r}}\, y), \text{etc}
\end{array}
$$

- where:

$$
\begin{array}{rcl}
z \,|^{\mathrm{sr}}{}_a \mathrm{NIL} & = & \mathrm{NIL} \\
z \,|^{\mathrm{sr}}{}_a (x + y) & = & z \,|^{\mathrm{sr}}{}_a x \; + \; z \,|^{\mathrm{sr}}{}_a y \\
z \,|^{\mathrm{sr}}{}_a b.y & = & \begin{cases} (a \cdot b).(z \,|^{\mathrm{l}}\, y + z \,|^{\mathrm{s}}\, y + z \,|^{\mathrm{r}}\, y) & (\text{if } a \cdot b \downarrow) \\ \mathrm{NIL} & (\text{otherwise}) \end{cases}
\end{array}
$$

Homomorphic definitions

Define

$$\bar{l} : \mathrm{ST} \to \mathrm{ST}^{\mathrm{ST}} \quad \bar{\mathrm{s}} : \mathrm{ST} \to \mathrm{ST}^{\mathrm{ST}} \quad \overline{\mathrm{sr}} : \mathrm{ST} \to \mathrm{ST}^{A \times \mathrm{ST}^{\mathrm{ST}}} \quad \bar{r} : \mathrm{ST} \to \mathrm{ST}^{\mathrm{ST}^{\mathrm{ST}}}$$

by

$$\bar{l}(a.x) = \lambda z.\, a.(\bar{l}(x)(z) + \bar{\mathrm{s}}(x)(z) + \bar{r}(z)(\bar{l}(x) + \bar{\mathrm{s}}(x)))$$

$$\bar{\mathrm{s}}(a.x) = \lambda z.\, \overline{\mathrm{sr}}_a(z)(\lambda v.\, \bar{l}(x)(v) + \bar{\mathrm{s}}(x)(v) + \bar{r}(v)(\bar{l}(x) + \bar{\mathrm{s}}(x)))$$

$$\overline{\mathrm{sr}}_a(b.y) = \lambda f. \begin{cases} (a \cdot b).(f(y)) & (\text{if } a \cdot b \downarrow) \\ \mathrm{NIL} & (\text{otherwise}) \end{cases}$$

$$\bar{r}(a.y) = \lambda f.\, a.(f(y) + \bar{r}(y)(f))$$

then put

$$x \,|^{\mathrm{l}}\, y = \bar{l}(x)(y) \quad x \,|^{\mathrm{s}}\, y = \bar{\mathrm{s}}(x)(y) \quad x \,|^{\mathrm{r}}\, y = \bar{r}(y)(\bar{l}(x) + \bar{\mathrm{s}}(x))$$

$$x \,|^{\mathrm{sr}}{}_a\, y = \overline{\mathrm{sr}}_a(y)(\lambda v : \mathrm{ST}.\, x \,|^{\mathrm{l}}\, v + x \,|^{\mathrm{s}}\, v + x \,|^{\mathrm{r}}\, v)$$

Prospects

- Can generalise the CCS deconstructors to all free algebras $T_{\mathrm{CCS}}(X)$, eg:

$$| : T_{\mathrm{CCS}}(X) \times T_{\mathrm{CCS}}(Y) \longrightarrow T_{\mathrm{CCS}}(X \times Y)$$

- To some extent can use other theories for CCS such as Milner's for $(+, \mathrm{NIL}, \tau)$.
- Prospect I: a principled combination of process algebra and functional programming.
- Examples: CSP (with van Glabbeek); INRIA join calculus; pi-calculus (Stark).
- Questions: Operational semantics? Logic?
- Prospect II: integration of process calculus theory with the theory of effects.