The power of Lambda calculus and Types

Exercises for EWSCS 2016

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In each exercise, the number of † 's indicates the level.

1. † (In this exercise, $=$ should be read as $=_{\mathsf{CL}}$.) Use the *definable abstraction in CL* to construct in each item a term P of CL with as few variables as possible such that:

$$
\begin{aligned}
P\,y &= y\,x &(1)\\
P\,x\,y &= y\,x &(2)\\
P\,x\,y &= y &(3)\\
P\,x &= x\,(x\,x) &(4)\\
P\,x\,y &= x\,x\,x &(5)\\
P\,x\,y &= x\,y\,y &(6)\\
P\,x\,y &= x\,(y\,y). &(7)
\end{aligned}
$$

2. † † (In this exercise, $=$ should be read as $=_{\mathsf{CL}}$.) Use the Fixed point theorem for CL to construct in each item a term Q of CL without variables such that:

$$
\begin{aligned}
Q &= Q\,\mathbf{K} &(8)\\
Q\,x &= x\,Q &(9)\\
Q &= Q\,\mathbf{I}\,\mathbf{I} &(10)\\
Q\,x &= Q\,x\,x. &(11)
\end{aligned}
$$

3. † † If we add to CL the axiom $P = Q$, and we can then prove that $U = V$, we write

$$P = Q \vdash U = V.$$

Show that

(a) $\mathbf{I} = \mathbf{K} \vdash x = y$
$\mathbf{I} = \mathbf{S} \vdash x = y$
$\mathbf{K} = \mathbf{S} \vdash x = y.$

(b) We assume that $\nvdash x = y$, or in a different notation $x \neq_{\mathsf{CL}} y$. (This is true, but we don't prove it.) From this, we immediately get, using (a), that

$$\mathbf{I} \neq \mathbf{K},\ \mathbf{I} \neq \mathbf{S},\ \mathbf{K} \neq \mathbf{S}.$$

Prove that $\mathbf{KK} \neq \mathbf{K}$.

(c) Prove that there are no terms F, G (without variables) such that

$$F(x\,y) = x,\ G(x\,y) = y.$$

[Hint. From the outcome of a computation $P\,A$ one cannot see what the original argument and the function have been. For example, $0 \times 1 = 0 \times 2$ en $2^2 = \sqrt{16}$.]

4. † except for (v) which is † † . (In this exercise, $=$ should be read as $=_{\beta}$.) Write down for each item a term $F \in \Lambda$, with a minimal number of variables, satisfying:

$$
\begin{array}{rrcl}
\text{(i)} & F\,x &=& x\,(x\,x)\\
\text{(ii)} & F\,x &=& x\,\mathbf{I}\\
\text{(iii)} & F\,x\,y &=& y\,x\\
\text{(iv)} & F\,x &=& y\,x\\
\text{(v)} & F\,x &=& x\,F\,\mathbf{I}.
\end{array}
$$

Use the Fixed Point Theorem in (v)

5. † † Let $W \equiv \lambda xy.x\,y\,y$. Draw $\mathcal{G}(W\,W\,W)$.
[Hint. This graph consists of exactly four terms.]

6. † † † (Exercise of [Jan Willem Klop])
Let $Y = L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L$, where

$$L = \lambda abcdefghijklmnopqstuvwxyzr.r\,(t\,h\,i\,s\,i\,s\,a\,f\,i\,x\,e\,d\,p\,o\,i\,n\,t\,c\,o\,m\,b\,i\,n\,a\,t\,o\,r).$$

Show that for all $F \in \Lambda$ one has $Y\,F = F\,(Y\,F)$. (So Y is a fixed point combinator.)

7. † † † Show that Y is a fixed point of **SI** implies that Y is a fixed point combinator.

8. (a) † † Use the fixed-point combinator to write down precisely a lambda term M such that

$$M\,x = x\,M\,x.$$

Can you make it satisfy $M\,x \twoheadrightarrow x\,M\,x$?

(b) † (Turing) Consider the term $\Theta := (\lambda xy.y(x\,x\,y))\,(\lambda xy.y(x\,x\,y))$. Show that Θ is a *reducing fixed point combinator*, that is:

$$\Theta\,F \twoheadrightarrow F(\Theta\,F) \text{ for all } F.$$

9. Remember that we can represent a natural number n as the lambda term $\mathbf{c}_n$, its so called *Church numeral*:

$$\mathbf{c}_n \equiv \lambda fa.f^n a,$$

where $f^n a$ denotes n-fold application of f on a. Define

$$A_\times \equiv \lambda nmfa.n(mf)a.$$

(a) † Show that $A_\times \mathbf{c}_2 \mathbf{c}_3 = \mathbf{c}_6$;

(b) † † Show that $A_\times \mathbf{c}_n \mathbf{c}_m = \mathbf{c}_{n\times m}$.

10. Remember the data type of (binary) trees from the lecture. Define the map f on trees as follows: $f(t)$ is obtained from t by replacing everywhere l by $j\,l\,(j\,l\,l)$.

(a) † Draw the trees t_1, t_2, $f(t_1)$ and $f(t_2)$, with $t_1 := j\,l\,l$, $t_2 := j\,(j\,l\,l)\,l$.

(b) Give the representation as a λ-term of the trees in (i).

(c) † † Construct a λ-term F such that for all trees t

$$F\ulcorner t\urcorner = \ulcorner f(t)\urcorner.$$

11. Consider the function h that counts the number of nodes in a tree (where we count a leaf also as a node), so

$$h(j\,l\,l) = 3$$
$$h(j\,l\,(j\,l\,l)) = 5$$

(a) † Write down a recursive definition for h, that is fill in

$$h(l) = \ldots$$
$$h(j\,t_1\,t_2) = \ldots h(t_1) \ldots h(t_2) \ldots$$

(b) † † Construct a term H that λ-defines h, that is

$$H\ulcorner t\urcorner = \ulcorner h(t)\urcorner$$

for all trees t.

12. (a) † † Given is the data type Nat with z : Nat, s : Nat → Nat.
Write down the codes (following Böhm, Guerrini, Piperno) of

$$2 = s(sz),\ 3 = s(s(sz)).$$

(b) † † Predecessor on Nat can be defined recursively:

$$\begin{array}{rcl} p(0) & = & 0 \\ p(n+1) & = & n. \end{array}$$

Using the theory you learned, construct a term P of the form $\langle\langle B_1, B_2\rangle\rangle$, to act on codes of Nat, such that

$$\begin{array}{rcl} P(\ulcorner z \urcorner) & = & \ulcorner 0 \urcorner \\ P(\ulcorner sn \urcorner) & = & \ulcorner n \urcorner. \end{array}$$

(c) † Verify $P\ulcorner 3 \urcorner = \ulcorner 2 \urcorner$.

13. (a) † Given is Tree, the data type with

$$\texttt{l} : \texttt{Tree}, \texttt{j} : \texttt{Tree}^2 \to \texttt{Tree}.$$

Write down the codes (following BGP) of

$$t_1 = j\,(j\,l\,l)\,l \quad \text{and} \quad t_2 = j\,l\,(j\,l\,l).$$

(b) † † Write down a λ-term $F = \langle\langle D_1, D_2\rangle\rangle$ (to act on codes of Tree) such that

$$\begin{array}{rcl} F\ulcorner l \urcorner & = & l \\ F\ulcorner j\,t\,s \urcorner & = & \ulcorner j\,t\,(j\,t\,s) \urcorner. \end{array}$$

(c) † Verify for the F you found that indeed $F\ulcorner j\,l\,(j\,l\,l) \urcorner = \ulcorner j\,l\,(j\,l\,(j\,l\,l)) \urcorner$.

14. † Check the statement on Slide 10 of the course slides, that

$$\begin{array}{rcl} H(\texttt{Var}\,x) & =_\beta & A_1\,x\,H \\ H(\texttt{App}\,x\,y) & =_\beta & A_2\,x\,y\,H \\ H(\texttt{Abs}\,x) & =_\beta & A_3\,x\,H \end{array}$$

if we take $H = \langle\langle B_1, B_2, B_3\rangle\rangle$ with

$$\begin{array}{rcl} B_1 & := & \lambda x\,z.A_1 x\langle z\rangle \\ B_2 & := & \lambda x\,y\,z.A_2 xy\langle z\rangle \\ B_3 & := & \lambda x\,z.A_3 x\langle z\rangle. \end{array}$$

and Var, App and Abs as on the slides. (Verify 2 of the equations for H.)

15. † † † Show that there is no term F such that

$$F(M\,N) =_\beta N \text{ for all terms } M, N.$$

16. Remember the definitions of $\mathbf{T} := \lambda xy.x (\equiv \mathbf{K})$ and $\mathbf{F} = \lambda xy.y (=_\beta \mathbf{KI})$.

(a) † † Construct a λ-term G such that

$$\begin{array}{rcl} G\ulcorner x \urcorner & = & \mathbf{T} \\ G\ulcorner PQ \urcorner & = & \mathbf{F} \\ G\ulcorner \lambda x.P \urcorner & = & \mathbf{F}. \end{array}$$

(b) †† Construct a λ-term V such that

$$\begin{aligned} V\ulcorner x\urcorner &= \mathbf{T} \\ V\ulcorner PQ\urcorner &= V\ulcorner P\urcorner \\ V\ulcorner \lambda x.P\urcorner &= \mathbf{F}. \end{aligned}$$

17. In the previous exercise you were asked to give terms completely. Now we just need show these terms exists, using the Recursion Theorem II of the slides.

(a) † Show that there is a λ-term G that checks whether a term is an abstraction, that is:

$$\begin{aligned} G\ulcorner x\urcorner &= \mathbf{F} \\ G\ulcorner PQ\urcorner &= \mathbf{F} \\ G\ulcorner \lambda x.P\urcorner &= \mathbf{T}. \end{aligned}$$

(b) ††† Show that there is a term R such that

$$\begin{aligned} R\ulcorner M\urcorner &= \mathbf{T} \text{ if } M \text{ is a redex} \\ R\ulcorner M\urcorner &= \mathbf{F} \text{ if } M \text{ is not a redex} \end{aligned}$$

(A redex is a term of the shape $(\lambda x.P)Q$.)

(c) ††† Show that there is a λ-term Norm that checks if a term is in *normal form*, so

$$\begin{aligned} \text{Norm}\ulcorner M\urcorner &= \mathbf{T} \text{ if } M \text{ is in normal form} \\ \text{Norm}\ulcorner M\urcorner &= \mathbf{F} \text{ if } M \text{ is not in normal form} \end{aligned}$$

(A term is *in normal form* if it contains no redex.)

(d) † You may need to define a conjunction & on Booleans first satisfying:

$$\begin{aligned} \&\ \mathbf{T}\,\mathbf{T} &= \mathbf{T} \\ \&\ \mathbf{T}\,\mathbf{F} &= \mathbf{F} \\ \&\ \mathbf{F}\,\mathbf{T} &= \mathbf{F} \\ \&\ \mathbf{F}\,\mathbf{F} &= \mathbf{F} \end{aligned}$$

18. (a) ††† Show that there is no term H' satisfying

$$\begin{aligned} H'\ulcorner M\urcorner\ulcorner N\urcorner &= \mathbf{T} \text{ if } M\,N \text{ has a normal form} \\ H'\ulcorner M\urcorner\ulcorner N\urcorner &= \mathbf{F} \text{ if } M\,N \text{ has no normal form} \end{aligned}$$

Hint: Show that, if H' exists, then we can also define the term B that solves the "blank tape" problem. (See course slides.)

(b) ††† Show that there is no term T satisfying

$$\begin{aligned} T\ulcorner M\urcorner &= \mathbf{T} \text{ if } M\,N \text{ has a normal form for all } N \\ T\ulcorner M\urcorner &= \mathbf{F} \text{ if } M\,N \text{ has no normal form for some } N \end{aligned}$$

Hint: Again, we can reduce B to T.

19. †††† Show that there is a term C that contracts the *left-most redex* in a term M, so

$$C\ulcorner M\urcorner = \ulcorner N\urcorner$$

if N arises from M by contracting the left-most redex, if M contains a redex, and otherwise $N \equiv M$.