Online Algorithms
Lecture 3

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EWSCS, Palmse, March 2020

A Brief Review of Bin Packing

Bin packing

- Input: Sequence of **items** $a_1, \ldots, a_n \in [0, 1]$.
- Output: Assign into **bins** of size 1.
- Objective: Minimize the number of bins.

Complexity results

- It is NP-hard to decide if $\mathrm{OPT}(I) = 2$.
  Thus it is NP-hard to approximate with ratio $< 3/2$.
- There exists a $3/2$-approximation algorithm.
- There exists an asymptotic approximation scheme.
  I.e., in polynomial time we can pack the items into $(1 + \varepsilon)\mathrm{OPT}(I) + 1$ bins.
- In poly time we can pack into $\mathrm{OPT}(I) + \log(\mathrm{OPT}(I))$ bins.

Performance measures

Absolute approximation ratio

For each instance I, the algorithm gives

$$ALG(I) \leq R \cdot \mathrm{OPT}(I)$$

Asymptotic approximation ratio

There exists a constant C such that for each instance I, the algorithm gives

$$ALG(I) \leq R \cdot \mathrm{OPT}(I) + C$$

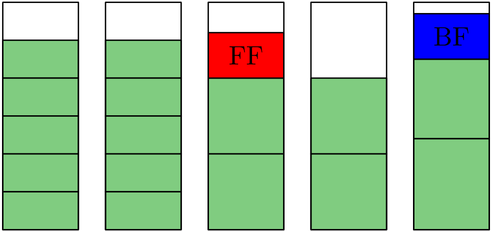
Two online algorithms

First Fit

Packs items one by one, always into the **first** bin where it fits.
Opens a new bin only when necessary.

Best Fit

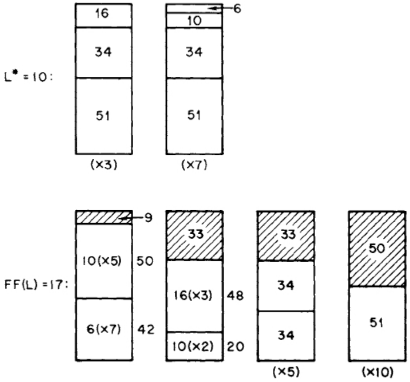
Packs items one by one, always into the **most full** bin where it fits.
Opens a new bin only when necessary.



Known Results

Classics [Ullman, 1971][Garey et al, 1973]

The asymptotic approximation ratio of both First Fit and Best Fit is equal to 1.7. More precisely, $\mathrm{FF}, \mathrm{BF} \le \lceil 1.7 \cdot \mathrm{OPT} \rceil$.
Example with $\mathrm{FF} = \mathrm{BF} = 17$ and $\mathrm{OPT} = 10$.



Bin size is 101.

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Example with $\mathrm{FF} = \mathrm{BF} = 17 \cdot k$ and $\mathrm{OPT} = 10 \cdot k + 1$.

Absolute ratio [Simchi-Levi, 1994]

$\mathrm{FF}, \mathrm{BF} \leq 1.75 \cdot \mathrm{OPT}$.

Recent improvements [Xia,Tan, 2010][Boyar et al, 2012][Neméth, 2011]

$\mathrm{FF} \leq 1.7 \cdot \mathrm{OPT} + 0.7$ and $\mathrm{FF} \leq 1.7119 \cdot \mathrm{OPT}$.
Uses analysis of cases with small OPT.

First Fit Decreasing [Johnson, 1973][Dósa, 2007]

$\mathrm{FFD} \leq \frac{11}{9} \cdot \mathrm{OPT} + \frac{2}{3}$, and this is optimal for all values of OPT.

Main Technique

Classical technique: **Weight functions**.

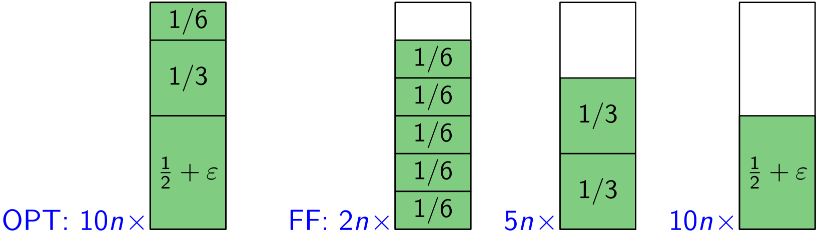
Find a weight of items such that

- Each bin in OPT has weight ≤ 1.7.
- Each bin in FF (BF) has weight ≥ 1 on average.

Combine weight functions with **amortized analysis**.

Idealized example for First Fit

Assume that the algorithm cannot have bins of size exactly 1.



Essentially, this can be achieved by changing the item sizes by a small amount.

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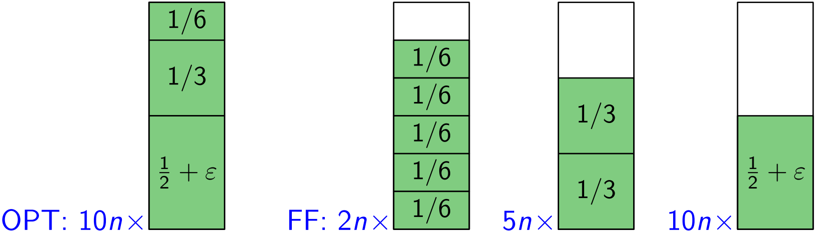

What should be the weights?

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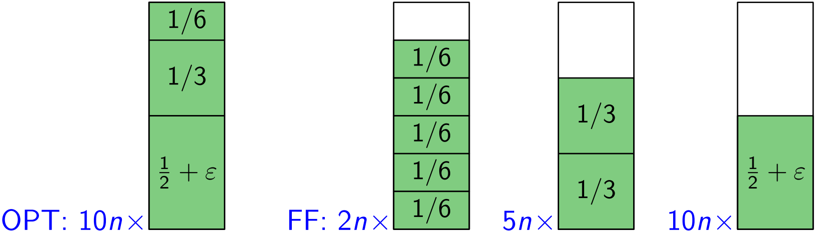

$w(1/6) = 0.2 \quad w(1/3) = 0.5 \quad w(\frac{1}{2}+\varepsilon) = 1$

The weight function

- Weight: **Scaled size** plus a **bonus.**
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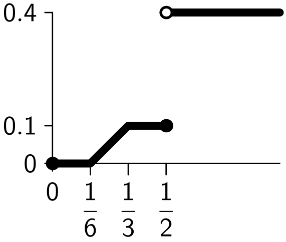
- $b(a) =$

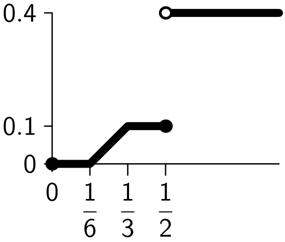
$$\begin{cases} 0 & \text{if } a \le \frac{1}{6} \\ \frac{3}{5}\left(a - \frac{1}{6}\right) & \text{if } a \in [\frac{1}{6}, \frac{1}{3}] \\ 0.1 & \text{if } a \in [\frac{1}{3}, \frac{1}{2}) \\ 0.4 & \text{if } a > \frac{1}{2} \end{cases}$$



Offline bins

Each bin (a set of items of size ≤ 1) contains **bonus items**:

- either no item of size $> 1/2$ and at most 5 items with bonus at most 0.1 each (actually the total is ≤ 0.3),
- or one item of size $> 1/2$ and at most 2 items with bonus at most 0.1 total.
- Thus the **total bonus** is at most 0.5;



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- Thus the **total bonus** is at most 0.5;
- the **total scaled size** is at most 1.2;
- the **total weight** is at most 1.7.

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- Bins with an item of size $> 1/2$ have weight ≥ 1.

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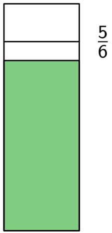
- There is at most one bin of size $\leq 1/2$.
- There is at most one bin of size $\leq 2/3$ with at least two items.
- Bins of size $\geq 5/6$ have weight ≥ 1.
- Bins with an item of size $> 1/2$ have weight ≥ 1.
- For the remaining bins with sizes in $(2/3, 5/6)$ we use **amortization**.
  We show that the scaled size of a bin plus the bonus of the **following** such bin is ≥ 1.

Amortization

For each bin with size in $(2/3, 5/6)$, at least two items, and no item $> 1/2$ we show that the scaled size of this bin plus the bonus of the **following** such bin is ≥ 1.



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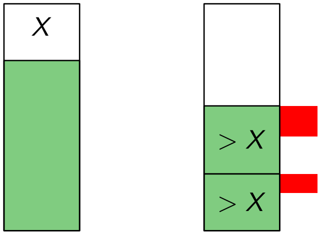

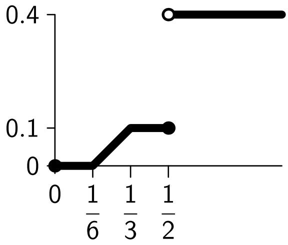

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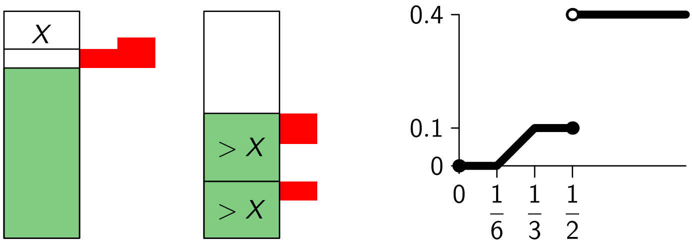

The result

$$\mathrm{FF}(I) - 3 < w(I) \leq 1.7 \cdot \mathrm{OPT}(I)$$

Theorem

First Fit has **asymptotic** approximation ratio 1.7.

$$\mathrm{FF}(I) < 1.7 \cdot \mathrm{OPT}(I) + 3$$

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Theorem

First Fit has **asymptotic** approximation ratio 1.7.

$$FF(I) < 1.7 \cdot OPT(I) + 3$$

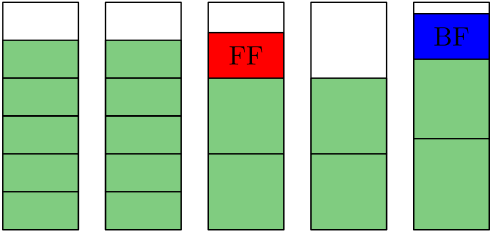
With some more work...

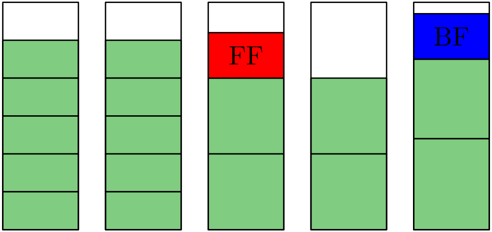
Theorem

First Fit and **Best Fit** have **absolute** approximation ratio 1.7.

$$FF(I), BF(I) \le 1.7 \cdot OPT(I)$$

First Fit vs. Best Fit





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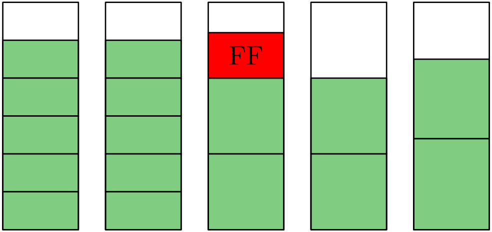

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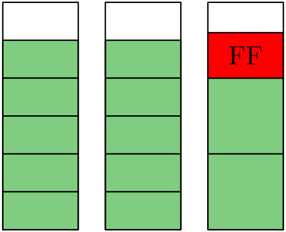

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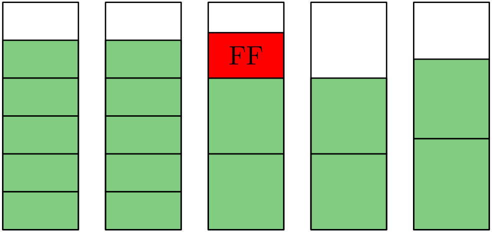
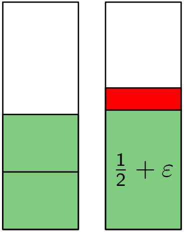


First Fit is a special case of Best Fit.

- Suppose we have a bad example for First Fit. No item fits into any previous bin.
- Reorder the instance so that first the items from the first bin arrive, then the items from the next bin, etc.
- The resulting First Fit and Best Fit packings are the same.

Best Fit revisited

Best Fit can create packings very different from First Fit packings.



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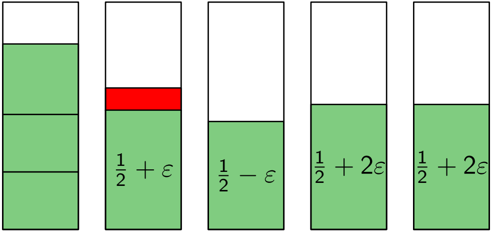

- We need some structural properties of Best Fit.
- Some remaining cases are difficult.

Other good algorithms

2-Bounded Space Best Fit (BBF_2)

Packs items as Best Fit into open bins.
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All of these are asymptotically 1.7-competitive.

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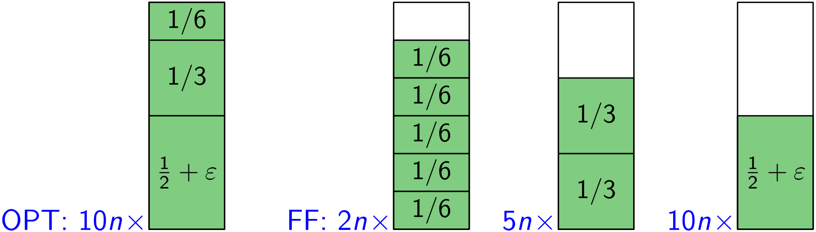
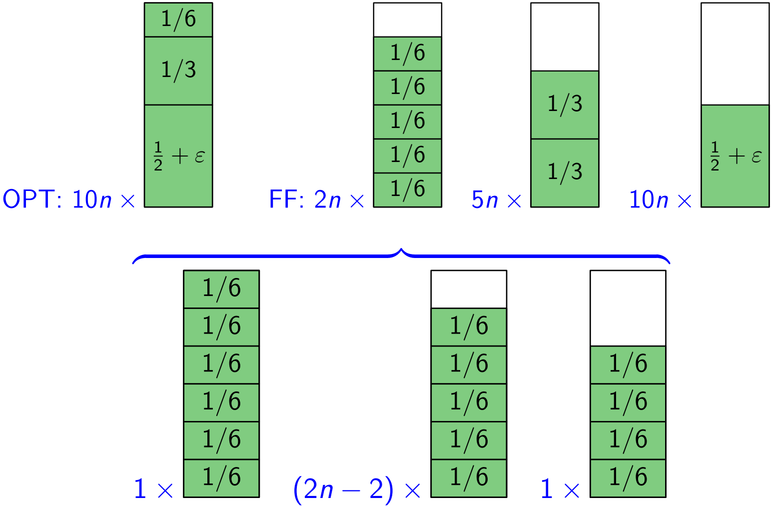
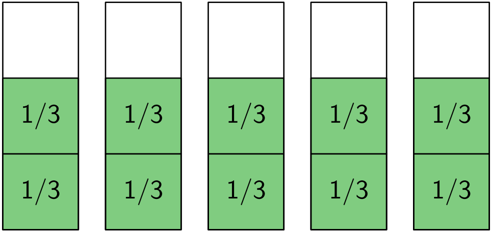


This can be achieved by changing the item sizes by a small amount and allowing OPT one extra bin.

Improved example



Optimal absolute ratio



- We need to avoid bins with two items and size about $2/3$.

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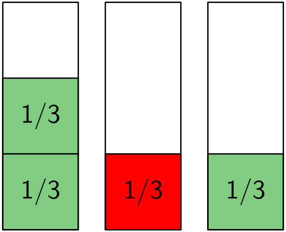

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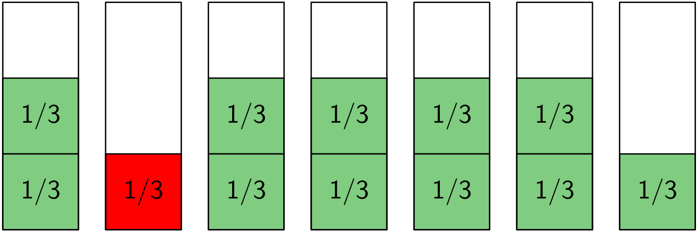

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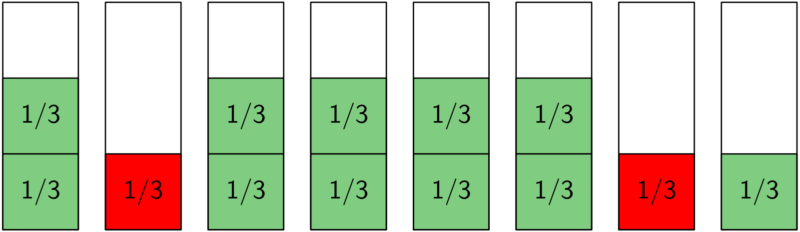

- We need to avoid bins with two items and size about 2/3.
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- Also instead of 6th, 10th, 14th, . . . , such bin
- "such bin" is actually a more technical notion.

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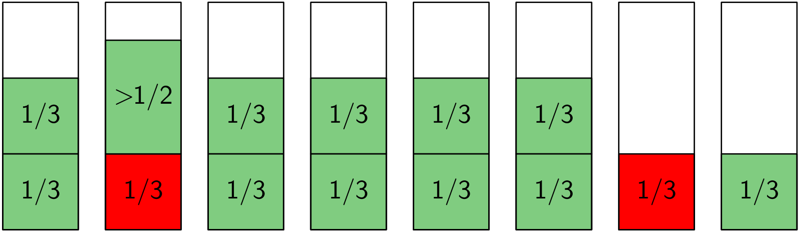

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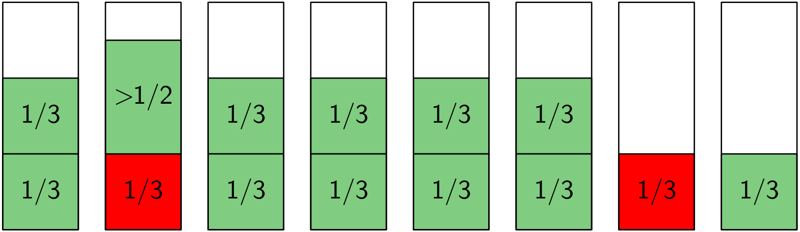
- Instead of 2nd such bin, we split the items and make a special bin.
- Also instead of 6th, 10th, 14th, ..., such bin
- "such bin" is actually a more technical notion.
- The special bins are reserved for large items.

Optimal absolute ratio



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Theorem

The approximation ratio of the algorithm is $5/3$ and this is optimal among online algorithms.

Three cases:

- No special bins: Relatively easy, like FF with only a few $1/3$'s.
- All special bins have a large item: Weight functions similar to the analysis of FF.
- Some special bin has no large item: All single large item bins have size much bigger than $1/2$ (actually $> 5/8$).
  Some small cases are tedious (and tight).

Tightness of the algorithm

Suppose that online algorithm ALG has ratio smaller than $5/3$.
Use this instance:

- 6 items of size $1/7$
  $\mathrm{OPT} = 1$, thus $\mathrm{ALG} = 1$.

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- 6 items of size $1/7$
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- 6 items of size $1/3 + \varepsilon$
  $\mathrm{OPT} = 3$, thus $\mathrm{ALG} = 4$.

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  $\text{OPT} = 1$, thus $\text{ALG} = 1$.
- 6 items of size $1/3 + \varepsilon$
  $\text{OPT} = 3$, thus $\text{ALG} = 4$.
- 6 items of size $1/2 + \varepsilon$
  $\text{OPT} = 6$, $\text{ALG} = 4 + 6 = 10$, a contradiction.

Bounded-space algorithms

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k-bounded-space BestFit [Csirik,Johnson,2001]

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- For $k \geq 2$, k-bounded-space BestFit has asymptotic approximation ratio 1.7.

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Optimal online algorithm [Lee,Lee,1985]

- There exists a family of k-bounded-space online algorithms whose asymptotic approximation ratio approaches $h_\infty \approx 1.691$ as $k \to \infty$.
- No k-bounded-space algorithm has a smaller asymptotic approximation ratio than $h_\infty \approx 1.691$.

Example – k-bounded space

- Input:

$$m \times \left(\tfrac{1}{7} + \varepsilon\right), \quad m \times \left(\tfrac{1}{3} + \varepsilon\right), \quad m \times \left(\tfrac{1}{2} + \varepsilon\right), \quad \text{where } m >> k$$

- k-bounded-space BestFit

$$\frac{m}{6} \times \left[\, 6\times \; \tfrac{1}{7} + \varepsilon \,\right] \qquad \frac{m}{2} \times \left[\, \tfrac{1}{3} + \varepsilon, \; \tfrac{1}{3} + \varepsilon \,\right] \qquad m \times \left[\, \tfrac{1}{2} + \varepsilon \,\right]$$

- Optimum:

$$m \times \left[\, \tfrac{1}{7} + \varepsilon, \; \tfrac{1}{3} + \varepsilon, \; \tfrac{1}{2} + \varepsilon \,\right]$$

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- Input:

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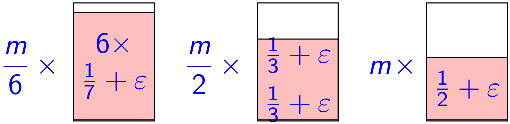
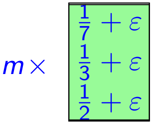
- k-bounded-space BestFit or any k-bounded-space algorithm:



- Optimum:



Open problems

- Does the 1.7 absolute approximation ratio hold for **Almost Any Fit algorithms?** And for **3-Bounded Space Best Fit?**

- The best lower and upper bounds on the **asymptotic** approximation ratio are 1.54278 and 1.58889.