Online Algorithms
Lecture 4

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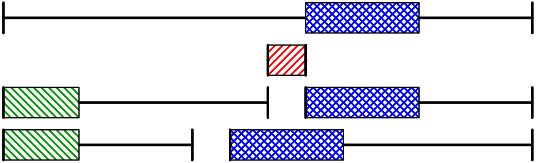
A puzzle



- Move boxes within their ranges.

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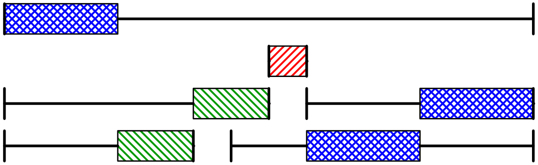

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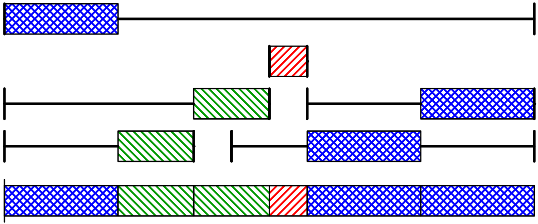

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- Align them so that they do not overlap vertically.
- Is this easy (in P) or difficult (NP-hard)?

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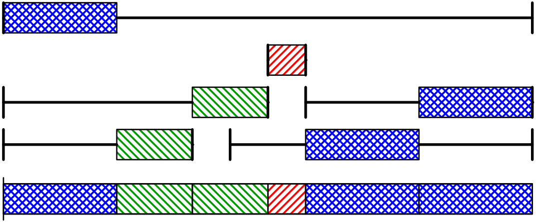

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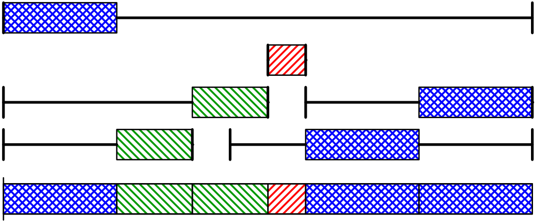

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Solved — Elffers, de Weerdt

Easy for two sizes 1 and p
NP-hard for any other pair of sizes
(including 2 and 4!)

Throughput scheduling

- Environment: One or more machines.
- Input: Jobs with length p_j, release time r_j, deadline d_j, and weight w_j. (Parameters are integers.)
- Output: Each job is assigned to a machine for a subinterval of $[r_j, d_j)$ of length p_j or rejected. No overlaps.
- Objective: Maximize the number (weight) of the completed jobs.

This talk

- Online algorithms.
- Usually a single machine.
- Either jobs of equal length ($p_j = p$) and no weights
- or jobs of unit length ($p_j = 1$) with weights.

Online scheduling

- At time r_j, the other parameters of the job become known.
- At each time, if a machine is idle, the algorithm may decide to start a job.

Competitive ratio

An algorithm A is R-competitive if for every instance I

- $OPT(I) \leq R \cdot A(I)$ for a deterministic algorithm, or
- $OPT(I) \leq R \cdot E[A(I)]$ for a randomized algorithm.

Jobs of equal length

Setting

- Equal lengths of jobs ($p_j = p$).
- No weights.
- Single machine.

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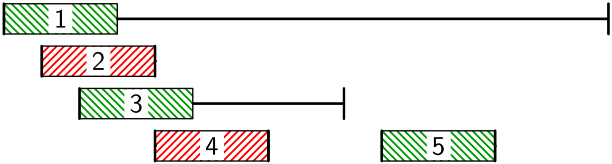
Outline

1. Offline problem is polynomial.
2. Greedy algorithms are 2-competitive.
3. Lower bounds.
4. A better randomized algorithm.
5. Generalizations, variants.

Greedy algorithms

GREEDY: If idle, start an arbitrary job.



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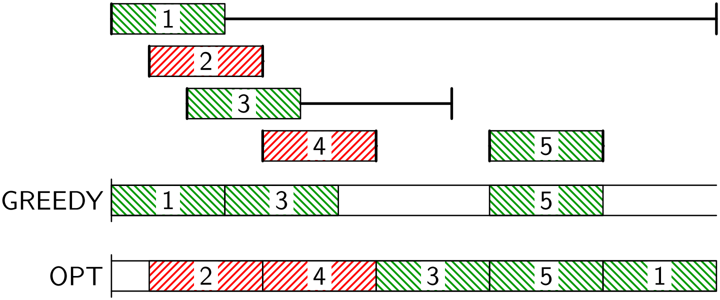

- Any such algorithm is 2-competitive.

Greedy algorithms

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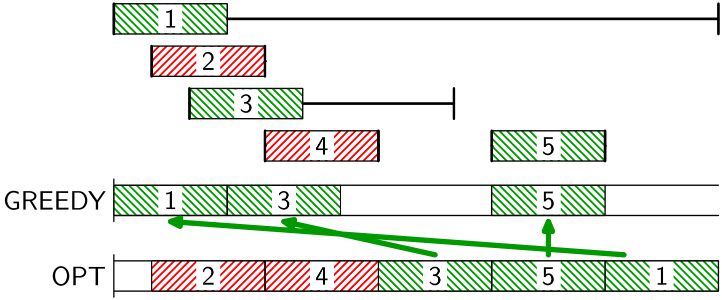

Charging scheme – GREEDY is 2-competitive

- Charge (map) a job in OPT to itself in GREEDY, if scheduled.

Greedy algorithms

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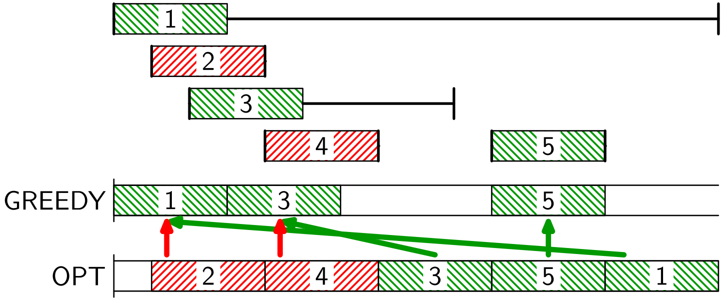
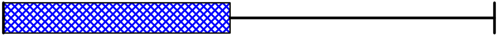


Charging scheme – GREEDY is 2-competitive

- Charge (map) a job in OPT to itself in GREEDY, if scheduled.
- Otherwise charge a job that OPT starts at t to the job GREEDY runs at t.

Lower bounds



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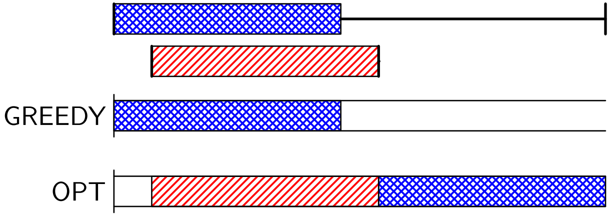

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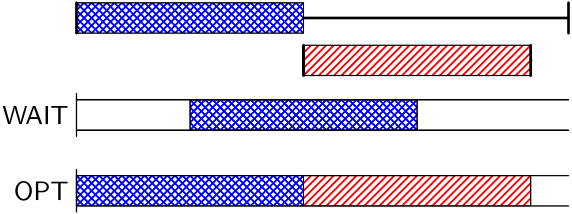
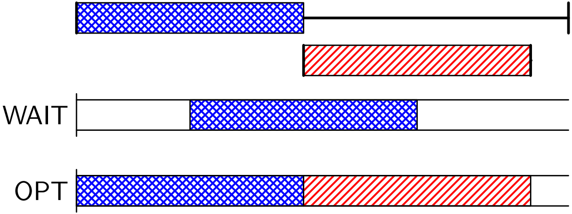


- No deterministic algorithm is better than 2-competitive.

Lower bounds



- No deterministic algorithm is better than 2-competitive.
- No randomized algorithm is better than $4/3$-competitive. (For one of the two instances, on average, runs at most 1.5 jobs out of 2.)

A barely random algorithm I

- Generate two schedules, A and B. Flip a coin to choose one of them.

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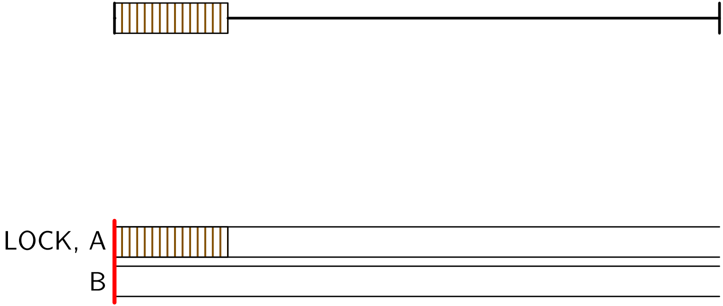
- A and B are produced by two identical processes using a common lock.

- If the machine is idle (in A or B) and the set of pending jobs is not flexible (idling for time p would lose some job), start the most urgent job.
- If the machine is idle (in A or B) and the set of pending jobs is flexible (idling for time p does no harm):
  - If the lock is available, acquire it, start the most urgent job and release the lock after the job is completed.
  - Otherwise stay idle.

A barely random algorithm II



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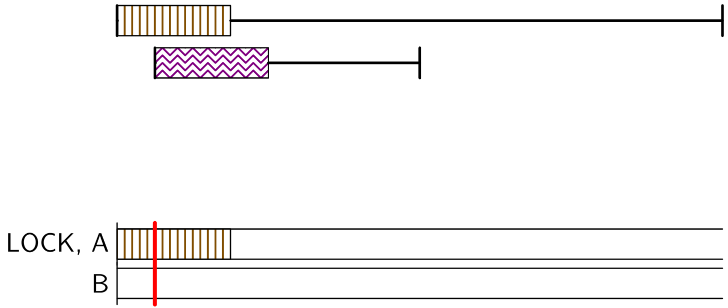

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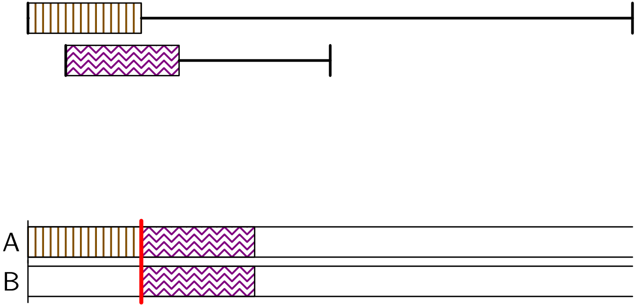

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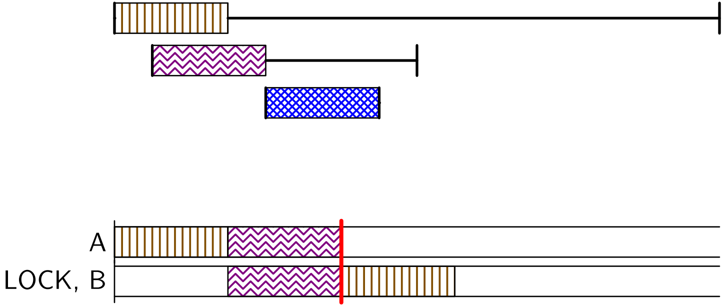

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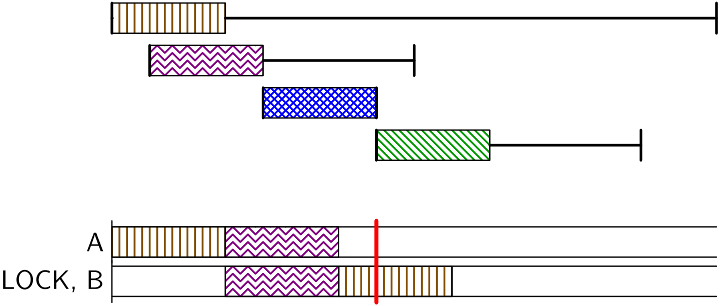

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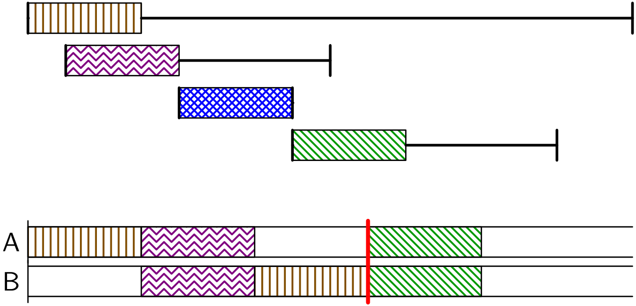

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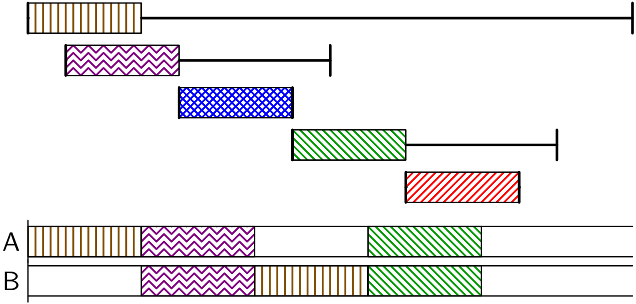

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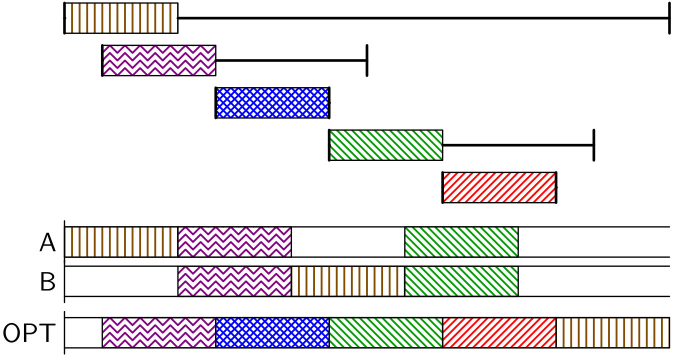

A barely random algorithm III

- Analyzed by a more complex charging scheme.
- Each job in OPT charges $1/2$, $1/3$, or $1/6$ to itself or to the job running at the same time in A and B.
- Each job in A or B is charged at most $5/6$.

A barely random algorithm III

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Theorem

This algorithm is $5/3$-competitive.

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Theorem

This algorithm is $5/3$-competitive.

Open problem

Find a randomized algorithm with the optimal competitive ratio.

More machines

Parallel machines make the problem easier!

Results

- For 2 machines, there is a $3/2$-competitive deterministic algorithm and this is optimal.
- For m machines, there is an R-competitive deterministic algorithm with $R \to e/(e-1) \approx 1.58$ for $m \to \infty$.
- The lower bound approaches $6/5$ for $m \to \infty$.

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Open problem

Decrease the gap for $m \to \infty$.

Unit time jobs with weights

Setting

- Unit length of jobs ($p_j = 1$).
- General weights.
- Single machine.

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Outline

1. Offline problem is easy (matching).
2. Greedy algorithm is 2-competitive.
3. A better randomized algorithm.
4. A better deterministic algorithm.
5. Generalizations, variants.

Motivation and variants

Forwarding packets in network switches

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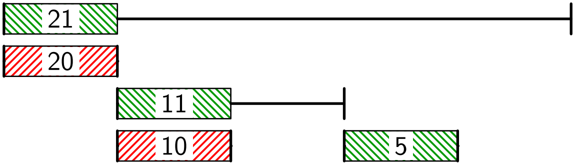
Forwarding packets in network switches

Restricted scenarios

- 2-bounded: Some packets may wait a single step, some packets not at all. ($d_j \leq r_j + 2$)
- Agreeable deadlines: $r_j < r_k$ implies $d_j \leq d_k$.
- Weighted queues: The deadlines are not known, only their order.
- Limited number of weights.

Greedy algorithm

GREEDY: If idle, start a pending job with the maximal weight.



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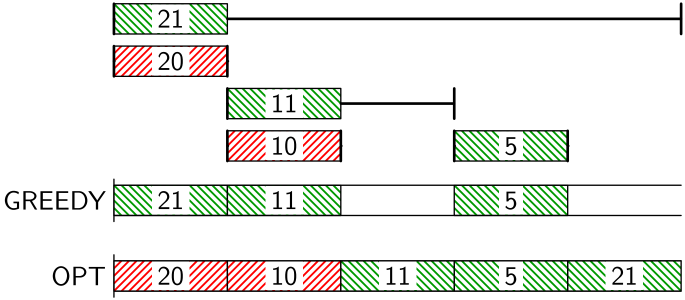

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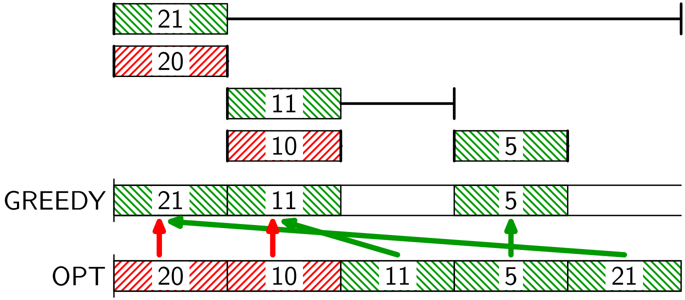

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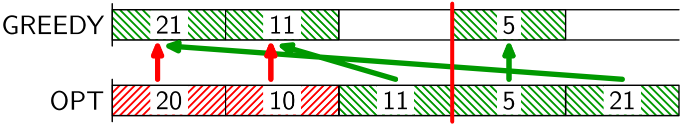
A randomized algorithm

- At each time, pick uniformly random real $x \in (-1, 0)$.
- Let h be the largest weight of a pending job.
- Among all the pending jobs with $w_j \geq e^x \cdot h$, schedule a job with the earlieast deadline.

Theorem

This algorithm is $e/(e-1) \approx 1.58$-competitive.

A potential function



How much "money" we need at a given time and configuration?

We earn $R \cdot w_j$ for running a job and pay w_j if OPT runs a job.

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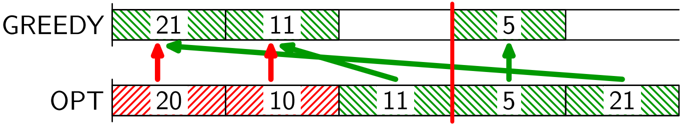

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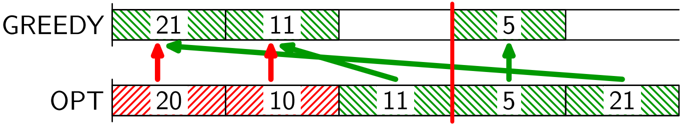

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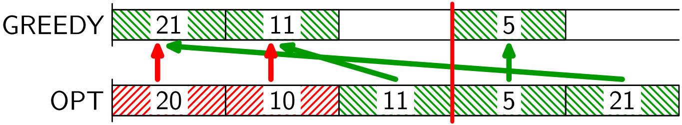

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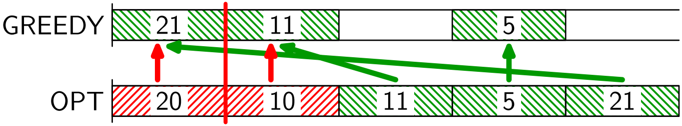

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This algorithm is $e/(e-1) \approx 1.58$-competitive. This is optimal against the adaptive online adversary. I.e., it is optimal among the algorithms analyzed using a potential.

Deterministic algorithms I

Charging scheme

Alternating heavy and urgent packets eventually leads to 1.939-competitive algorithm.

Deterministic algorithms I

Charging scheme

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Potential function

Can be used to give a 1.828-competitive algorithm.

Deterministic algorithms II

Modifying the optimal schedule

At each step, the configuration of the optimal schedule is made identical with that of the online algorithm, with some advantage to the optimum:

- Schedule a job and keep it pending,
- Schedule two jobs,
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Can be used to give a $\phi \approx 1.618$-competitive algorithm for instances with agreeable deadlines.

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Weighted queues

There exists a 1.897-competitive algorithm.

Lower bounds

2-bounded instances

- The $\phi \approx 1.618$-competitive deterministic algorithm is optimal.
- There exists a 1.25-competitive randomized algorithm and this is optimal.

No other lower bounds for the general problem are known.

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Open problem

Is the general problem harder than the 2-bounded case?

New deterministic algorithm

Algorithm LESSGREEDY(ϕ)

- Schedule packet $p \in \mathcal{P}$ maximizing $\phi \cdot w_p + w(\mathcal{Q}_p)$
  - $\mathcal{P}$: the optimal current schedule
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- Very elegant algorithm . . . but *not* ϕ-competitive

Optimal deterministic algorithm

Amortization Techniques

1. Increasing weights in the algorithm's plan
   - Algorithm's future profit *may* get higher
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Optimal deterministic algorithm

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Theorem

There is a ϕ-competitive deterministic algorithm.

Open Problems

$m \geq 1$ packets sent in each step

- Our algorithm is $\phi \approx 1.618$-competitive for any $m \geq 1$
- Best upper bound tends to $\frac{e}{e-1} \approx 1.58$

Randomized algorithms

- Gap between 1.25 and $\frac{e}{e-1} \approx 1.58$
- Algorithms that see only an ordering of the weights, not their values

Memoryless algorithms

- Is there a lower bound $> \phi$ for memoryless algorithms?
- What is the ratio of $\textsc{LessGreedy}(\alpha)$? (Schedule $p \in \mathcal{P}$ max. $\alpha \cdot w_p + w(\mathcal{Q}_p)$)