Fixed Points on Partial Randomness

Kohtaro Tadaki*
Research and Development Initiative, Chuo University
JST CREST
1–13–27 Kasuga, Bunkyo-ku, Tokyo 112-8551, Japan
tadaki@kc.chuo-u.ac.jp

Abstract

Algorithmic information theory (AIT, for short) is a theory of program-size and algorithmic randomness. One of the primary concepts of AIT is the *Kolmogorov complexity* $K(s)$ of a finite binary string s, which is defined as the length of the shortest binary program for a universal decoding algorithm to output s. In this paper, we report on a quite new type of fixed point in computer science, called a *fixed point on partial randomness*. In the research of AIT, it is important to consider the notion of the *compression rate* of a real T, which is defined as the real $\lim_{n\to\infty} K(T\restriction_n)/n$, where $T\restriction_n$ is the first n bits of the base-two expansion of T. The notion of the *partial randomness* of a real is a stronger representation of the compression rate. Our fixed point theorems on partial randomness give sufficient conditions for a real $T \in (0,1)$ to satisfy that the partial randomness of T equals to T and therefore the compression rate of T equals to T. The fixed point theorems are obtained in the framework of the statistical mechanical interpretation of AIT developed by our works [K. Tadaki, Local Proceedings of CiE 2008, pp. 425–434, 2008] and [K. Tadaki, Proceedings of LFCS'09, Springer's LNCS, vol. 5407, pp. 422–440, 2009]. As an original contribution of this paper, we present a simple and self-contained proof of the fixed point theorem on partial randomness.

1 Introduction and Summary

Algorithmic information theory (AIT, for short) is a framework to apply information-theoretic and probabilistic ideas to recursive function theory. One of the primary concepts of AIT is the *Kolmogorov complexity* (or *program-size complexity*) $K(s)$ of a finite binary string s, which is defined as the length of the shortest binary input for a universal decoding algorithm U, called an *optimal prefix-free machine*, to output s. By the definition, $K(s)$ is thought to represent the amount of randomness contained in s, which cannot be captured in a computational manner. In particular, the notion of Kolmogorov complexity plays a crucial role in characterizing the *randomness* of an infinite binary string, or equivalently, a real.

In this paper, we report on a quite new type of fixed point in computer science, called a *fixed point on partial randomness* [15, 16]. In the research of AIT, it is important to consider the notion of the *compression rate* of a real α, which is defined as the real $\lim_{n\to\infty} K(\alpha\restriction_n)/n$, where $\alpha\restriction_n$ is the first n bits of the base-two expansion of α. The notion of the *partial randomness* of a real is a stronger representation of the compression rate. *The fixed point theorems on partial randomness* give a sufficient condition for a real $T \in (0,1)$ to be a fixed point on partial randomness, i.e., to satisfy that the partial randomness of T equals to T.[1] One form of the fixed point theorems on partial randomness is presented as follows: For every $T \in (0,1)$, if $Z(T)$ is a computable real, then the partial randomness of T equals to T, and therefore the compression rate of T equals to T, i.e.,

$$\lim_{n\to\infty} \frac{K(T\restriction_n)}{n} = T, \tag{1}$$

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[1] The fixed point theorems on partial randomness were called fixed point theorems on compression rate in [15].

where $Z(T)$ is defined by

$$Z(T) := \sum_{U(p)\text{ is defined}} 2^{-\frac{|p|}{T}}. \tag{2}$$

Intuitively, we might interpret the meaning of (1) as follows: Consider imaginarily a file of infinite size whose content is

"The compression rate of this file is 0.100111001……"

When this file is compressed, the compression rate of this file actually equals to 0.100111001……, as the content of this file says. This situation is self-referential and forms a fixed point.

The fixed point theorems on partial randomness are obtained in the framework of the statistical mechanical interpretation of AIT developed by the works [15, 16]. In the development of the interpretation, we introduced the notion of *thermodynamic quantities at temperature* T, such as partition function $Z(T)$, free energy $F(T)$, energy $E(T)$, and statistical mechanical entropy $S(T)$, into AIT. These quantities are real functions of a real argument $T > 0$, and are defined based on the halting set of the optimal prefix-free machine U, like in (2). We proved that if T is a computable real with $0 < T < 1$ then, for each of the thermodynamic quantities at temperature T, the partial randomness of its value equals to T, and therefore the compression rate of its value equals to T. Thus, the temperature T plays a role as the partial randomness of all the thermodynamic quantities in the statistical mechanical interpretation of AIT. Furthermore, we showed that this situation holds for the temperature T itself, which is a thermodynamic quantity of itself in thermodynamics and statistical mechanics. Namely, we proved the fixed point theorems on partial randomness presented above, where the computability of $Z(T)$ gives a sufficient condition for $T \in (0,1)$ to be a fixed point on partial randomness. In addition, we showed that the computability of each of the remaining thermodynamic quantities $F(T)$, $E(T)$, and $S(T)$ also gives the sufficient condition. Moreover, based on the "statistical mechanical" relation $F(T) = -T\log_2 Z(T)$, we showed that the computability of $F(T)$ gives completely different fixed points from the computability of $Z(T)$.

The paper is organized as follows. We begin in Section 2 with some preliminaries to AIT and partial randomness. In Section 3, we give the definitions of the thermodynamic quantities in AIT and investigate their properties on partial randomness. The fixed point theorems on partial randomness are presented in Section 4. As an original contribution of this paper, in Section 5 we present a simple and self-contained proof of the fixed point theorem on partial randomness by $Z(T)$ stated above. We refer the reader to our original papers [15, 16] for the detail of this work and its related topics.

2 Preliminaries

We first review some basic notation and definitions which will be used in this paper. $\mathbb{N} = \{0,1,2,3,\dots\}$ is the set of natural numbers, and $\mathbb{N}^+$ is the set of positive integers. $\mathbb{Q}$ is the set of rationals, and $\mathbb{R}$ is the set of reals. $\{0,1\}^* = \{\lambda,0,1,00,01,10,11,000,\dots\}$ is the set of finite binary strings, where λ denotes the *empty string*. For any $s \in \{0,1\}^*$, $|s|$ is the *length* of s. A subset V of $\{0,1\}^*$ is called *prefix-free* if no string in V is a prefix of another string in V. It is easy to show that every prefix-free set $V \subset \{0,1\}^*$ satisfies the so-called *Kraft inequality* $\sum_{s\in V} 2^{-|s|} \le 1$. For any partial function f, the domain of definition of f is denoted by $\operatorname{dom} f$. We write "r.e." instead of "recursively enumerable." A real α is called *computable* if there exists a total recursive function $f\colon \mathbb{N}^+ \to \mathbb{Q}$ such that $|\alpha - f(n)| < 1/n$ for all $n \in \mathbb{N}^+$. For any $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}^+$, we denote by $\alpha\restriction_n \in \{0,1\}^*$ the first n bits of the base-two expansion of $\alpha - \lfloor\alpha\rfloor$ with infinitely many zeros, where $\lfloor\alpha\rfloor$ is the greatest integer less than or equal to α. For example, in the case of $\alpha = 5/8$, $\alpha\restriction_6 = 101000$.

In what follows we concisely review some definitions and results of AIT. For the detail, see [3, 4, 8, 5]. A *prefix-free machine* is a partial recursive function $M\colon \{0,1\}^* \to \{0,1\}^*$ such that $\operatorname{dom} M$ is a prefix-free set. For any prefix-free machine M and any $s \in \{0,1\}^*$, $K_M(s)$ is defined by $K_M(s) = \min\{\,|p| \mid p \in \{0,1\}^* \ \&\ M(p) = s\,\}$ (may be ∞). A prefix-free machine U is said to be *optimal* if for each prefix-free machine M there exists $c \in \mathbb{N}$ with the following property; if $p \in \operatorname{dom} M$, then there is $q \in \operatorname{dom} U$ for which $U(q) = M(p)$ and $|q| \le |p| + c$. There exists an optimal prefix-free machine. We choose a particular optimal prefix-free machine U as the standard one for use, and define $K(s)$ as $K_U(s)$, which is referred to as the *Kolmogorov complexity* of s or the *program-size complexity* of s. It follows that for every prefix-free machine M there exists $c \in \mathbb{N}$ such that, for every $s \in \{0,1\}^*$,

$$K(s) \le K_M(s) + c. \tag{3}$$

Based on this we can show that there exists $c \in \mathbb{N}$ such that, for every $s \neq \lambda$,

$$K(s) \le |s| + 2\log_2 |s| + c. \tag{4}$$

Using (3), it is also easy to show that, for every partial recursive function $\Psi\colon \{0,1\}^* \to \{0,1\}^*$, there exists $c \in \mathbb{N}$ such that, for every $s \in \operatorname{dom}\Psi$,

$$K(\Psi(s)) \le K(s) + c. \tag{5}$$

An element of $\operatorname{dom} U$ is called a *program for* U.

Let T be an arbitrary real with $0 < T \le 1$. In the work [14], we introduced several notions of the partial randomness of a real by parameterizing the notions of randomness of a real by a real T, as follows. Let $\alpha \in \mathbb{R}$. We say that α is *weakly Chaitin T-random* if there exists $c \in \mathbb{N}$ such that $Tn - c \le K(\alpha\restriction_n)$ for all $n \in \mathbb{N}^+$. On the other hand, we say that α is *T-compressible* if $K(\alpha\restriction_n) \le Tn + o(n)$, which is equivalent to $\limsup_{n\to\infty} K(\alpha\restriction_n)/n \le T$. Thus, if α is weakly Chaitin T-random and T-compressible, then

$$\lim_{n\to\infty} \frac{K(\alpha\restriction_n)}{n} = T. \tag{6}$$

The left-hand side of (6) is referred to as the *compression rate* of a real α in general. Note, however, that (6) does not necessarily imply that α is weakly Chaitin T-random. Thus, the notion of partial randomness is a stronger representation of the compression rate. We say that α is *Chaitin T-random* if $\lim_{n\to\infty} K(\alpha\restriction_n) - Tn = \infty$. Obviously, if α is Chaitin T-random, then α is weakly Chaitin T-random. However, in 2005 Reimann and Stephan [9] showed that, in the case of $T < 1$, the converse does not necessarily hold.

3 Thermodynamic Quantities in AIT

We introduce the notion of thermodynamic quantities into AIT in the following manner.

In statistical mechanics, the partition function $Z_{\mathrm{sm}}(T)$, free energy $F_{\mathrm{sm}}(T)$, energy $E_{\mathrm{sm}}(T)$, and entropy $S_{\mathrm{sm}}(T)$ at temperature T are given as follows:

$$\begin{aligned} Z_{\mathrm{sm}}(T) &= \sum_{x\in X} e^{-\frac{E_x}{k_{\mathrm{B}}T}}, & F_{\mathrm{sm}}(T) &= -k_{\mathrm{B}}T \ln Z_{\mathrm{sm}}(T), \\ E_{\mathrm{sm}}(T) &= \frac{1}{Z_{\mathrm{sm}}(T)} \sum_{x\in X} E_x e^{-\frac{E_x}{k_{\mathrm{B}}T}}, & S_{\mathrm{sm}}(T) &= \frac{E_{\mathrm{sm}}(T) - F_{\mathrm{sm}}(T)}{T}, \end{aligned} \tag{7}$$

where X is a complete set of energy eigenstates of a quantum system and E_x is the energy of an energy eigenstate x. The constant k_{B} is called *the Boltzmann Constant*, and the ln denotes the natural logarithm.[2]

[2] For the thermodynamic quantities in statistical mechanics, see e.g. Chapter 16 of [1]. To be precise, the partition function is not a thermodynamic quantity but a statistical mechanical quantity.

We introduce the notion of thermodynamic quantities into AIT by performing Replacements 1 below for the thermodynamic quantities (7) in statistical mechanics.

Replacements 1.

(i) Replace the complete set X of energy eigenstates x by the set $\operatorname{dom} U$ *of all programs p for U.*

(ii) Replace the energy E_x of an energy eigenstate x by the length $|p|$ of a program p.

(iii) Set the Boltzmann Constant k_{B} to $1/\ln 2$. □

For that purpose, we first choose a particular recursive enumeration $p_1, p_2, p_3, p_4, \dots$ of the infinite r.e. set $\operatorname{dom} U$ as the standard one for use throughout the rest of this paper.[3] Then, motivated by the formulae (7) and taking into account Replacements 1, we introduce the notion of thermodynamic quantities into AIT as follows.

Definition 3.1 (thermodynamic quantities in AIT, [15])**.** *Let T be any real with $T > 0$.*

(i) The partition function $Z(T)$ at temperature T is defined as $\lim_{k\to\infty} Z_k(T)$ *where*

$$Z_k(T) = \sum_{i=1}^{k} 2^{-\frac{|p_i|}{T}}.$$

(ii) The free energy $F(T)$ at temperature T is defined as $\lim_{k\to\infty} F_k(T)$ *where*

$$F_k(T) = -T \log_2 Z_k(T).$$

(iii) The energy $E(T)$ at temperature T is defined as $\lim_{k\to\infty} E_k(T)$ *where*

$$E_k(T) = \frac{1}{Z_k(T)} \sum_{i=1}^{k} |p_i| 2^{-\frac{|p_i|}{T}}.$$

(iv) The statistical mechanical entropy $S(T)$ at temperature T is defined as $\lim_{k\to\infty} S_k(T)$ *where*

$$S_k(T) = \frac{E_k(T) - F_k(T)}{T}.$$

□

Since $\operatorname{dom} U$ is prefix-free, we first see that the Kraft inequality $Z(1) \le 1$ holds. The real $Z(1)$ is precisely a Chaitin Ω number introduced by Chaitin [3]. For every $T \in (0,1]$, $Z(T)$ converges and $Z(T) \le 1$ since $2^{-|p_i|/T} \le 2^{-|p_i|}$.

Theorem 3.2 (properties of $Z(T)$ and $F(T)$, [14, 15])**.** *If T is a computable real with* $0 < T \le 1$*, then each of $Z(T)$ and $F(T)$ converges and is weakly Chaitin T-random and T-compressible.* □

Theorem 3.3 (properties of $E(T)$ and $S(T)$, [15])**.** *If T is a computable real with* $0 < T < 1$*, then each of $E(T)$ and $S(T)$ converges and is Chaitin T-random and T-compressible.* □

The above two theorems show that if T is a computable real with $T \in (0,1)$ then the temperature T equals to the partial randomness (and therefore the compression rate) of the values of all the thermodynamic quantities in Definition 3.1. Note that the weak Chaitin T-randomness in Theorems 3.2 is strengthen to the Chaitin T-randomness in Theorems 3.3.

[3] Actually, the enumeration $\{p_i\}$ can be chosen quite arbitrarily, and the results of this paper are independent of the choice of $\{p_i\}$. This is because the sum $\sum_{i=1}^{k} 2^{-|p_i|/T}$ and $\sum_{i=1}^{k} |p_i| 2^{-|p_i|/T}$ in Definition 3.1 are positive term series and converge as $k \to \infty$ for every $T \in (0,1)$.

4 Fixed Point Theorems on Partial Randomness

In statistical mechanics or thermodynamics, among all thermodynamic quantities one of the most typical thermodynamic quantities is temperature itself. Inspired by this fact in physics and by the observation in the previous section that the temperature T equals to the partial randomness of the values of the thermodynamic quantities in the statistical mechanical interpretation of AIT, the following question arises naturally: Can the partial randomness of the temperature T equal to the temperature T itself in the statistical mechanical interpretation of AIT ? This question is rather self-referential. However, we can answer it affirmatively in the following form.

Theorem 4.1 (fixed point theorem by $Z(T)$, [15])**.** *For every $T \in (0,1)$, if $Z(T)$ is computable, then T is weakly Chaitin T-random and T-compressible, and therefore $\lim_{n\to\infty} K(T\restriction_n)/n = T$.* □

Theorem 4.1 is just a *fixed point theorem on partial randomness*, where the computability of the value $Z(T)$ gives a sufficient condition for a real $T \in (0,1)$ to be a fixed point on partial randomness. Thus, the above observation that the temperature T equals to the partial randomness of the values of the thermodynamic quantities in the statistical mechanical interpretation of AIT is further confirmed. In addition, we can show that fixed point theorems of the same form as Theorem 4.1 hold also for the free energy $F(T)$, energy $E(T)$, and statistical mechanical entropy $S(T)$, as follows. Thus we confirm the above observation much further.

Theorem 4.2 (fixed point theorem by $F(T)$, [16])**.** *For every $T \in (0,1)$, if $F(T)$ is computable then T is weakly Chaitin T-random and T-compressible.* □

Theorem 4.3 (fixed point theorem by $E(T)$, [16])**.** *For every $T \in (0,1)$, if $E(T)$ is computable then T is Chaitin T-random and T-compressible.* □

Theorem 4.4 (fixed point theorem by $S(T)$, [16])**.** *For every $T \in (0,1)$, if $S(T)$ is computable then T is Chaitin T-random and T-compressible.* □

Note that the weak Chaitin T-randomness of T in Theorems 4.1 is strengthen to the Chaitin T-randomness of T in Theorems 4.3 and 4.4. Theorem 4.1 will be proved in Section 5 in a self-contained manner. Since the function $Z(T)$ of T is monotonically increasing and continuous on $(0,1)$, and the set of all computable reals is dense in $\mathbb{R}$, the following theorem holds for the sufficient condition of Theorem 4.1. The exactly same theorem holds for each of $F(T)$, $E(T)$, and $S(T)$ also [16].

Theorem 4.5 ([15])**.** *The set $\{\, T \in (0,1) \mid Z(T)$ is computable$\,\}$ is dense in $(0,1)$.* □

Using the "statistical mechanical" relation $F(T) = -T\log_2 Z(T)$ we can show Theorem 4.6 below. Thus, the computability of $F(T)$ gives completely different fixed points from the computability of $Z(T)$. This implies that neither the computability of $Z(T)$ nor the computability of $F(T)$ is the necessary condition for $T \in (0,1)$ to be a fixed point on partial randomness at all.

Theorem 4.6 ([16])**.** *There does not exist $T \in (0,1)$ such that both $Z(T)$ and $F(T)$ are computable.* □

Using the property of T as a fixed point in Theorems 4.1, we can show the following.

Theorem 4.7 ([16])**.** *$S_a \cap S_b = \emptyset$ for any distinct computable reals $a, b \in (0,1]$, where $S_a = \{\, T \in (0,1) \mid Z(aT)$ is computable$\,\}$.*

Proof. Let $T \in (0,1)$, and let a be a computable real with $a \in (0,1]$. Suppose that $Z(aT)$ is computable. Then, by Theorem 4.1, $\lim_{n\to\infty} K((aT)\restriction_n)/n = aT$. Since $K((aT)\restriction_n) = K(T\restriction_n) + O(1)$ it follows that $\lim_{n\to\infty} K(T\restriction_n)/n = aT$. Thus, for every computable reals $a, b \in (0,1]$, if $S_a \cap S_b \neq \emptyset$ then $a = b$. □

As a corollary of Theorem 4.7, we have the following, for example.

Corollary 4.8 ([16]). *For every $T \in (0,1)$, if $Z(T)$ is computable, then $Z(T/n)$ is not computable for every $n \in \mathbb{N}^+$ with $n \geq 2$. In other words, for every $T \in (0,1)$, if the sum $\sum_{i=1}^{\infty} 2^{-|p_i|/T}$ is computable, then the corresponding power sum $\sum_{i=1}^{\infty} \left(2^{-|p_i|/T}\right)^n$ is not computable for every $n \in \mathbb{N}^+$ with $n \geq 2$.* □

5 The Proof of Theorem 4.1

As an original contribution of this paper, we present a simple and self-contained proof of Theorem 4.1 in what follows. We first recall the notion of right computable enumerability and left computable enumerability of a real. A real α is called *right computably enumerable* (*right-c.e.*, for short) if there exists a total recursive function $f\colon \mathbb{N}^+ \to \mathbb{Q}$ such that $\alpha \leq f(n)$ for all $n \in \mathbb{N}^+$ and $\lim_{n\to\infty} f(n) = \alpha$. Right-c.e. reals are also called *right-computable*. On the other hand, a real α is called *left computably enumerable* (*left-c.e.*, for short) if $-\alpha$ is right-c.e. Left-c.e. reals are also called *left-computable*. It is then easy to show the following theorem.

Theorem 5.1. *Let $\alpha \in \mathbb{R}$.*

(i) α is computable if and only if α is both right-c.e. and left-c.e.

(ii) α is right-c.e. if and only if the set $\{\, r \in \mathbb{Q} \mid \alpha < r \,\}$ is r.e. □

Theorem 4.1 follows immediately from Theorem 5.2, Theorem 5.3, and Theorem 5.4 below, as well as from Theorem 5.1 (i).

Theorem 5.2. *For every $T \in (0,1)$, if $Z(T)$ is right-c.e. then T is weakly Chaitin T-random.*

Proof. First, for each $k \in \mathbb{N}^+$ and each real $x > 0$, we define $W_k(x)$ as $\sum_{i=1}^{k} |p_i| 2^{-|p_i|/x}$. We show that, for each $x \in (0,1)$, $W_k(x)$ converges as $k \to \infty$. Let x be an arbitrary real with $x \in (0,1)$. Since $x < 1$, there is $l_0 \in \mathbb{N}^+$ such that $(\log_2 l)/l \leq 1/x - 1$ for all $l \geq l_0$. Then there is $k_0 \in \mathbb{N}^+$ such that $|p_i| \geq l_0$ for all $i > k_0$. Thus, we see that, for each $i > k_0$,

$$|p_i| 2^{-\frac{|p_i|}{x}} = 2^{-(\frac{1}{x} - \frac{\log_2 |p_i|}{|p_i|})|p_i|} \leq 2^{-|p_i|}.$$

Hence, for each $k > k_0$, $W_k(x) - W_{k_0}(x) = \sum_{i=k_0+1}^{k} |p_i| 2^{-|p_i|/x} \leq \sum_{i=k_0+1}^{k} 2^{-|p_i|} < Z(1)$. Therefore, since $\{W_k(x)\}_k$ is an increasing sequence of reals bounded to the above, it converges as $k \to \infty$, as desired. For each $x \in (0,1)$, we define a positive real number $W(x)$ as $\lim_{k\to\infty} W_k(x)$.

On the other hand, since $Z(T)$ is right-c.e. by the assumption, there exists a total recursive function $f\colon \mathbb{N}^+ \to \mathbb{Q}$ such that $Z(T) \leq f(m)$ for all $m \in \mathbb{N}^+$, and $\lim_{m\to\infty} f(m) = Z(T)$.

We choose a particular real t with $T < t < 1$. Then, for each $i \in \mathbb{N}^+$, using the mean value theorem we see that

$$2^{-\frac{|p_i|}{x}} - 2^{-\frac{|p_i|}{T}} < \frac{\ln 2}{T^2} |p_i| 2^{-\frac{|p_i|}{t}} (x - T)$$

for all $x \in (T,t)$. We then choose a particular $c \in \mathbb{N}$ with $W(t) \ln 2/T^2 \leq 2^c$. Here, the limit value $W(t)$ exists, since $0 < t < 1$. It follows that

$$Z_k(x) - Z_k(T) < 2^c (x - T) \tag{8}$$

for all $k \in \mathbb{N}^+$ and $x \in (T,t)$. We also choose a particular $n_0 \in \mathbb{N}^+$ such that $0.(T{\restriction_n}) + 2^{-n} < t$ for all $n \geq n_0$. Such n_0 exists since $T < t$ and $\lim_{n\to\infty} 0.(T{\restriction_n}) + 2^{-n} = T$. Since $T{\restriction_n}$ is the first n bits of the base-two expansion of T with infinitely many zeros, we then see that $T < 0.(T{\restriction_n}) + 2^{-n} < t$ for all $n \geq n_0$.

Now, given $T\restriction_n$ with $n \ge n_0$, one can find $k_0, m_0 \in \mathbb{N}^+$ such that $f(m_0) < Z_{k_0}(0.(T\restriction_n)+2^{-n})$. This is possible from $Z(T) < Z(0.(T\restriction_n)+2^{-n})$, $\lim_{k\to\infty} Z_k(0.(T\restriction_n)+2^{-n}) = Z(0.(T\restriction_n)+2^{-n})$, and the properties of f. It follows from $Z(T) \le f(m_0)$ and (8) that $\sum_{i=k_0+1}^{\infty} 2^{-|p_i|/T} = Z(T) - Z_{k_0}(T) < Z_{k_0}(0.(T\restriction_n)+2^{-n}) - Z_{k_0}(T) < 2^{c-n}$. Hence, for every $i > k_0$, $2^{-|p_i|/T} < 2^{c-n}$ and therefore $Tn - Tc < |p_i|$. Thus, by calculating the set $\{\, U(p_i) \mid i \le k_0 \,\}$ and picking any one finite binary string which is not in this set, one can then obtain an $s \in \{0,1\}^*$ such that $Tn - Tc < K(s)$.

Hence, there exists a partial recursive function $\Psi\colon \{0,1\}^* \to \{0,1\}^*$ such that $Tn - Tc < K(\Psi(T\restriction_n))$ for all $n \ge n_0$. Using (5), there is $c_\Psi \in \mathbb{N}$ such that $K(\Psi(T\restriction_n)) \le K(T\restriction_n) + c_\Psi$ for all $n \ge n_0$. Therefore, $Tn - Tc - c_\Psi < K(T\restriction_n)$ for all $n \ge n_0$. It follows that T is weakly Chaitin T-random. □

Theorem 5.3. *For every $T \in (0,1)$, if $Z(T)$ is right-c.e., then T is also right-c.e.*

Proof. Since $Z(T)$ is right-c.e., there exists a total recursive function $f\colon \mathbb{N}^+ \to \mathbb{Q}$ such that $Z(T) \le f(m)$ for all $m \in \mathbb{N}^+$, and $\lim_{m\to\infty} f(m) = Z(T)$. Thus, since $Z(x)$ is an increasing function of $x \in (0,1]$, we see that, for every $x \in \mathbb{Q}$ with $0 < x < 1$, $T < x$ if and only if there are $m, k \in \mathbb{N}^+$ such that $f(m) < Z_k(x)$. It follows from Theorem 5.1 (ii) that T is right-c.e. □

Theorem 5.4. *For every $T \in (0,1)$, if $Z(T)$ is left-c.e. and T is right-c.e., then T is T-compressible.*

Proof. For each $i \in \mathbb{N}^+$, using the mean value theorem we see that

$$2^{-\frac{|p_1|}{t}} - 2^{-\frac{|p_1|}{T}} > (\ln 2)\,|p_1|\,2^{-\frac{|p_1|}{T}}(t-T)$$

for all $t \in (T,1)$. We choose a particular $c \in \mathbb{N}^+$ such that $(\ln 2)\,|p_1|\,2^{-\frac{|p_1|}{T}} \ge 2^{-c}$. Then, it follows that

$$Z_k(t) - Z_k(T) > 2^{-c}(t-T) \tag{9}$$

for all $k \in \mathbb{N}^+$ and $t \in (T,1)$.

Since T is a right-c.e. real with $T < 1$ by the assumption, there exists a total recursive function $f\colon \mathbb{N}^+ \to \mathbb{Q}$ such that $T < f(l) < 1$ for all $l \in \mathbb{N}^+$, and $\lim_{l\to\infty} f(l) = T$. On the other hand, since $Z(T)$ is left-c.e. by the assumption, there exists a total recursive function $g\colon \mathbb{N}^+ \to \mathbb{Q}$ such that $g(m) \le Z(T)$ for all $m \in \mathbb{N}^+$, and $\lim_{m\to\infty} g(m) = Z(T)$. Let $\Omega = Z(1)$. By Theorem 3.2, $Z(1)$ is weakly Chaitin 1-random and therefore $Z(1) \notin \mathbb{Q}$. Thus, the base-two expansion of Ω is unique and contains infinitely many ones, and $0 < \Omega < 1$ in particular.

Given n and $\Omega\restriction_{\lceil Tn\rceil}$ (i.e., the first $\lceil Tn\rceil$ bits of the base-two expansion of Ω), one can find $k_0 \in \mathbb{N}^+$ such that $0.(\Omega\restriction_{\lceil Tn\rceil}) < \sum_{i=1}^{k_0} 2^{-|p_i|}$. This is possible since $0.(\Omega\restriction_{\lceil Tn\rceil}) < \Omega$ and $\lim_{k\to\infty} \sum_{i=1}^{k} 2^{-|p_i|} = \Omega$. It is then easy to see that $\sum_{i=k_0+1}^{\infty} 2^{-|p_i|} = \Omega - \sum_{i=1}^{k_0} 2^{-|p_i|} < 2^{-\lceil Tn\rceil} \le 2^{-Tn}$. Using the inequality $a^d + b^d \le (a+b)^d$ for any reals $a, b > 0$ and $d \ge 1$, it follows that

$$Z(T) - Z_{k_0}(T) = \sum_{i=k_0+1}^{\infty} 2^{-\frac{|p_i|}{T}} < 2^{-n}. \tag{10}$$

Note that $\lim_{l\to\infty} Z_{k_0}(f(l)) = Z_{k_0}(T)$. Thus, since $Z_{k_0}(T) < Z(T)$, one can then find $l_0, m_0 \in \mathbb{N}^+$ such that $Z_{k_0}(f(l_0)) < g(m_0)$. It follows from (10) and (9) that $2^{-n} > g(m_0) - Z_{k_0}(T) > Z_{k_0}(f(l_0)) - Z_{k_0}(T) > 2^{-c}(f(l_0) - T)$. Thus, $0 < f(l_0) - T < 2^{c-n}$. Let t_n be the first n bits of the base-two expansion of the rational number $f(l_0)$ with infinitely many zeros. Then, $|f(l_0) - 0.t_n| < 2^{-n}$. It follows from $|T - 0.(T\restriction_n)| < 2^{-n}$ that $|0.(T\restriction_n) - 0.t_n| < (2^c+2)2^{-n}$. Hence, $T\restriction_n = t_n, t_n \pm 1, t_n \pm 2, \dots, t_n \pm (2^c+1)$, where $T\restriction_n$ and t_n are regarded as a dyadic integer. Thus, there are still $2^{c+1}+3$ possibilities of $T\restriction_n$, so that one needs only $c+2$ bits more in order to determine $T\restriction_n$.

Thus, there exists a partial recursive function $\Phi\colon \mathbb{N}^+ \times \{0,1\}^* \times \{0,1\}^* \to \{0,1\}^*$ such that

$$\forall n \in \mathbb{N}^+ \quad \exists s \in \{0,1\}^* \quad |s| = c+2 \ \&\ \Phi(n, \Omega\restriction_{\lceil Tn\rceil}, s) = T\restriction_n\,.$$

It follows from (4) that $K(T\restriction_n) \le |\Omega\restriction_{\lceil Tn \rceil}| + o(n) \le Tn + o(n)$, which implies that T is T-compressible. □

6 Concluding Remarks

By a series of works of Ryabko [10, 11], Staiger [12, 13], Lutz [6], Tadaki [14], and Mayordomo [7] over the last two decades, the equivalence between the notion of *Hausdorff dimension* and the notion of compression rate by Kolmogorov complexity seems to be established at present. In particular, Tadaki [14] considered the equivalence between the notion of Hausdorff dimension and the notion of partial randomness as well as the compression rate. In the context of the subject of the equivalence, we can consider the notion of the *dimension* of an individual real in particular, and this notion plays a crucial role in the subject. Our fixed point theorems on partial randomness give sufficient conditions for a real $T \in (0,1)$ to satisfy that the dimension of T equals to T. Thus, it would be interesting if we could develop the subject of the equivalence further in a new direction based on the notion of fixed point on partial randomness.

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