Scattered Algebraic Linear Orderings

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Abstract

An algebraic linear ordering is a component of the initial solution of a first-order recursion scheme over the continuous categorical algebra of countable linear orderings equipped with the sum operation and the constant 1. Due to a general Mezei-Wright type result, algebraic linear orderings are exactly those isomorphic to the linear ordering of the leaves of an algebraic tree. Moreover, using a result of Courcelle together with a Mezei-Wright type result, we can show that the algebraic words are exactly those that are isomorphic to the lexicographic ordering of a deterministic context-free language. Algebraic well-orderings have been shown to be those well-orderings whose order type is less than ω^{ω^ω}. We prove that the Hausdorff rank of any scattered algebraic linear ordering is less than ω^ω.

1 Introduction

Fixed points and finite systems of fixed point equations occur in just about all areas of computer science. Regular and context-free languages, rational and algebraic formal power series, finite state process behaviors can all be characterized as (components of) canonical solutions (e.g., unique, least or greatest, or initial or final solutions) of systems of fixed point equations, or recursion schemes.

In this paper we consider systems fixed point equations over countable linear orderings. Consider for example the system

$$\begin{aligned} X &= X + Y + X \\ Y &= \mathbf{1} + Y \end{aligned}$$

where $+$ denotes the usual sum operation on linear orderings, and $\mathbf{1}$ is a singleton linear ordering. It has no solution among finite linear orderings, but it has many solutions among countable linear orderings. The second component of the simplest "canonical" solution is the ordering $\mathbb{N}$ of the nonnegative integers, whereas the first component is the ordering obtained from the ordering $\mathbb{Q}$ of the rationals by replacing each point with a copy of $\mathbb{N}$.

In the above "regular" system of equations, the unknowns X,Y range over linear orderings. More generally, in an "algebraic" or "first-order" scheme we allow unknowns ranging over functions, or rather, functors defined on linear orderings:

$$\begin{aligned} X &= Y(\mathbf{1}) \\ Y(x) &= Z(x) + Y(\mathbf{1} + x) \\ Z(x) &= Z(x) + x + Z(x) \end{aligned}$$

*Supported in part by grant no. K 75249 from the National Foundation of Hungary for Scientific Research.

Here, X ranges over linear orderings, while Y,Z range over functions (or more precisely, over functors) on linear orderings. The first component of the canonical solution of this system is the linear ordering $L_1 + L_2 + \ldots$, where for each $n > 0$, L_n is the linear ordering obtained from $\mathbb{Q}$ by replacing each point with the linear ordering $\mathbf{n}$, the n-fold sum of $\mathbf{1}$ with itself.

Regular linear orderings are a special case of the regular words (or arrangements) of Courcelle [9]. Regular words and linear orders were studied in [18, 16, 3, 4]. The study of algebraic words and linear orderings was initiated in [5]. As an application of a general Mezei-Wright type result [6], one obtains that a linear ordering is algebraic (regular) iff it is isomorphic to the linear ordering of the leaves of an algebraic (regular) tree. (See [10, 15] for the definition of algebraic and regular trees.)

In this paper, we first review the characterization of algebraic linear orderings by deterministic context-free languages. Then we show that the Hausdorff rank of every **scattered** algebraic linear ordering is less than ω^ω. This extends one direction of a result of [7] where it is shown that an ordinal is algebraic iff it is less than ω^{ω^ω}. As a consequence of our results, we also obtain that if a scattered linear ordering is isomorphic to the ordering of a deterministic context-free language, then its Hausdorff rank is less than ω^ω.

2 Basic Notions and Notation

2.1 Continuous Categorical Σ-Algebras

Suppose that $\Sigma = \bigcup_{n\geq 0} \Sigma_n$ is a ranked alphabet. A **categorical Σ-algebra** ([4, 5, 6]) $\mathscr{A}$ consists of a (small) category, also denoted $\mathscr{A}$ together with a functor $\sigma^{\mathscr{A}} : \mathscr{A}^n \to \mathscr{A}$, for each letter $\sigma \in \Sigma_n$, called the operation induced by σ. A morphism of categorical Σ-algebras is a functor which preserves the operations up to natural isomorphism.

We say that a categorical Σ-algebra $\mathscr{A}$ is **continuous** if it has initial object and colimits of ω-diagrams, moreover, the operations $\sigma^{\mathscr{A}}$ are continuous, i.e., they preserve colimits of ω-diagrams in each argument. Morphisms of continuous categorical Σ-algebras are continuous and preserve initial objects.

The notion of continuous categorical Σ-algebra generalizes the notion of continuous ordered Σ-algebra [14, 15], where the underlying category is a poset. Some examples of continuous categorical Σ-algebras are given below.

2.2 Linear Orderings

In this paper, a linear ordering $(W,<)$ is a **countable** set W equipped with a strict linear order relation $<$. (To force the collection of all words to be a small set, we may require that the underlying set of a linear ordering is a subset of a fixed set.) A **morphism** between linear orderings $(W,<) \to (V,<)$ is a function $W \to V$ which preserves the order relation (and is thus injective). The category **Lin** of linear orderings has as initial object the empty linear ordering denoted $\mathbf{0}$. Moreover, **Lin** has colimits of all ω-diagrams.

Let Δ be a ranked alphabet with $\Delta_2 = \{+\}$, $\Delta_0 = \{\mathbf{1}\}$ and $\Delta_n = \emptyset$ for all $n \notin \{0,2\}$. We turn **Lin** into a categorical Δ-algebra by interpreting the binary symbol $+$ as the usual **sum functor** $\mathbf{Lin}^2 \to \mathbf{Lin}$ and $\mathbf{1}$ as a singleton linear ordering. The sum functor maps a pair of linear orderings $(W_i,<_i)$, $i = 1,2$ to the linear ordering $(W_1 + W_2, <)$ whose underlying set is the disjoint union of W_1 and W_2 and such that the restriction of $<$ to W_i is $<_i$, for $i = 1,2$. The sum $h_1 + h_2$ of morphisms $h_i : W_i \to V_i$, $i = 1,2$ is defined so that it agrees with h_i on W_i, for $i = 1,2$. Equipped with these functors, **Lin** is a continuous categorical Δ-algebra.

2.3 Trees

Let Σ be any ranked set. An example of a continuous categorical Σ-algebra is the algebra T_Σ^∞ of all finite and infinite Σ-trees defined in the usual manner. This continuous categorical Σ-algebra is ordered, so that there is at most one morphism between any two trees. It is known that T_Σ^∞ is the *essentially unique* initial continuous categorical Σ-algebra. See [14, 15, 6] for more details.

3 Recursion Schemes

Definition 3.1. *A* ***recursion scheme*** *over Σ is a sequence E of equations*

$$\begin{aligned} F_1(v_1,\ldots,v_{k_1}) &= t_1 \\ &\vdots \\ F_n(v_1,\ldots,v_{k_n}) &= t_n \end{aligned} \tag{1}$$

where t_i is a term over the ranked alphabet $\Sigma \cup \mathscr{F}$ in the variables $v_1,\ldots,v_{k_i}$, for $i \in [n]$, where $\mathscr{F} = \{F_1,\ldots,F_n\}$. A recursion scheme is ***regular*** *if $k_i = 0$, for each $i \in [n]$.*

In the above definition, $\Sigma \cup \mathscr{F}$ is the ranked alphabet whose letters are the letters in Σ together with the letters in $\{F_1,\ldots,F_n\}$ where each F_i is of rank k_i.

In any continuous categorical Σ-algebra $\mathscr{A}$, any scheme E induces a continuous endofunctor $E^{\mathscr{A}}$ over the category

$$[\mathscr{A}^{k_1} \to \mathscr{A}] \times \ldots [\mathscr{A}^{k_n} \to \mathscr{A}]$$

where $[\mathscr{A}^k \to \mathscr{A}]$ denotes the category of all continuous functors $\mathscr{A}^k \to \mathscr{A}$. Since this category also has initial object and colimits of ω-diagrams, it has an **initial fixed point** $|E^{\mathscr{A}}|$ (cf. [1, 19]) which is unique up to isomorphism.

Definition 3.2. *Suppose that $\mathscr{A}$ is a continuous categorical Σ-algebra. We call a functor $f : \mathscr{A}^m \to \mathscr{A}$,* ***algebraic*** *if there is a recursion scheme E such that f is isomorphic to $|E|_1^A$, the first component of the above initial solution. When $m = 0$, f may be identified with an object of A, called an* ***algebraic object****. An object a in $\mathscr{A}$ is* ***regular*** *if there is a regular recursion scheme E such that a is isomorphic to $|E|_1^{\mathscr{A}}$.*

By applying the above notion to **Lin** and T_Σ^∞, we obtain the notions of algebraic and regular linear orderings, and algebraic and regular trees, respectively. Several characterizations of algebraic and regular trees can be found in [14, 10, 15]. For characterizations of regular linear orderings we refer to [9, 3]. Here we only mention the following fact.

Let Δ be the ranked alphabet defined above in Section 2.2. Then there is a unique morphism of categorical Δ-algebras $T_\Delta^\infty \to$ **Lin**, namely the **frontier map** mapping each tree to the linear ordering of its leaves. Due to a Mezei-Wright type result [6] we have:

Proposition 3.3. *A countable linear ordering is algebraic or regular iff it is isomorphic to the frontier of an algebraic or regular tree in T_Δ^∞.*

Actually the above fact holds for all ranked sets Σ such that Σ_0 is not empty and there is at least one $n > 1$ such that Σ_n is also not empty.

4 Representing Linear Orderings by Languages of Finite Words

Let A be an alphabet equipped with a fixed linear order relation that we extend to the lexicographic order $<_\ell$ of A^*, the set of (isomorphism types) of finite words. If $L \subseteq A^*$ is any language, then $(L, <_\ell)$ is a linear ordering. When A has two or more letters, then every countable linear ordering is isomorphic to a linear ordering $(L, <_\ell)$. We can also show that every **recursive** linear ordering is isomorphic to an ordering $(L, <_\ell)$, for some recursive language $L \subseteq A^*$.

Definition 4.1. *Call a linear ordering **context-free** (**deterministic context-free**, respectively) if it is isomorphic to a linear ordering $(L, <_\ell)$ for some context-free (deterministic context-free, respectively) language L over some alphabet A (or equivalently, over the 2-letter alphabet $\{0, 1\}$).*

Using Courcelle's characterization of algebraic trees by deterministic context-free languages from [10] together with Proposition 3.3, we have:

Proposition 4.2. *A linear ordering is algebraic iff it is deterministic context-free.*

There is a similar characterization of regular linear orderings using ordinary regular languages.

5 Scattered Algebraic Linear Orderings

A good treatment of linear orderings is [17]. Recall from [17] that a linear ordering is **scattered** if it has no subordering isomorphic to the ordering of the rationals.

Scattered (countable) linear orderings can be classified into a transfinite hierarchy. Let V_0 denote the empty linear ordering and the singleton linear orderings. When α is a nonzero ordinal, let V_α be the collection of all linear orderings that can be obtained from a subordering P of $\mathbb{Z}$, the ordering of the integers by replacing each point $x \in P$ with a linear ordering in V_{β_x} for some $\beta_x < \alpha$. By Hausdorff's theorem, a linear ordering is scattered iff it belongs to V_α for some (countable) ordinal α, and the least such ordinal is called the **Hausdorff rank** or **VD**-rank of the scattered linear ordering.

It is known (see [16, 2, 5]) that a well-ordering is regular iff its order type is less than ω^ω, or equivalently, when its Hausdorff rank is finite. Moreover, the Hausdorff rank of every scattered regular linear ordering is finite. In [7], it is shown that a well-ordering is algebraic iff its order type is less than that of the ordinal ω^{ω^ω}, or equivalently, when its Hausdorff rank is less than ω^ω.

The main result of this paper is:

Theorem 5.1. *The Hausdorff rank of any scattered algebraic linear ordering is less than ω^ω.*

Corollary 5.2. *The Hausdorff rank of any scattered deterministic context-free ordering is less than ω^ω.*

6 Conclusion and Open Problems

A hierarchy of recursion schemes was introduced in [11], see also [12, 13]. Here, we dealt with level 0 (regular schemes) and level 1 (algebraic or first-order schemes) of the hierarchy. In Theorem 5.1, we have shown that every scattered linear ordering definable by a level 1 scheme is of Hausdorff rank less than ω^ω, whereas it has been known that the Hausdorff rank of any scattered linear ordering definable by a recursion scheme of level 0 is less than ω. We conjecture that for each n, the Hausdorff rank of any scattered linear ordering definable by a level n scheme is less than $\Uparrow(\omega, n+1)$, a tower of $n+1$ ω's. If that conjecture is true, then it follows that an ordinal is definable by a level n scheme iff it is less than $\Uparrow(\omega, n+2)$, and thus an ordinal is definable in the hierarchy iff it is less than ε_0. (See also [8].)

In ordinal analysis of logical theories, the strength of a theory is measured by ordinals. For example, the proof theoretic ordinal of Peano arithmetic is ε_0. Here we have a similar phenomenon: we measure the strength of recursive definitions by ordinals, and we conjecture that the ordinals definable are exactly those less than ε_0.

Finally, we mention two open problems.

Problem 1. Is there a context-free linear order which is not a deterministic context-free linear order?

Problem 2. Characterize the context-free well orderings and scattered linear orderings.

References

[1] J. Adámek. Free algebras and automata realizations in the language of categories. *Comment. Math. Univ. Carolinae*, 15:589–602, 1974.

[2] S. L. Bloom and C. Choffrut. Long words: the theory of concatenation and ω-power. *Theor. Comput. Sci.*, 259(1–2):533–548, 2001.

[3] S. L. Bloom and Z. Ésik. Deciding whether the frontier of a regular tree is scattered. *Fund. Inform.*, 11:1–22, 2004.

[4] S. L. Bloom and Z. Ésik. The equational theory of regular words. *Inform. and Comput.*, 197(1–2):55–89, 2005.

[5] S. L. Bloom and Z. Ésik. Regular and algebraic words and ordinals. In T. Mossakowski et al., ed., *Proc. of 2nd Int. Conf. on Algebra and Coalgebra in Computer Science, CALCO 2007 (Bergen, Aug. 2007)*, v. 4624 of *Lect. Notes in Comput. Sci.*, pp. 1–15. Springer, 2007.

[6] S. L. Bloom and Z. Ésik. A Mezei-Wright theorem for categorical algebras. *Theor. Comput. Sci.*, to appear.

[7] S. L. Bloom and Z. Ésik. Algebraic ordinals. Submitted for publication.

[8] L. Braud. Unpublished paper.

[9] B. Courcelle. Frontiers of infinite trees. *Theor. Inform. and Appl.*, 12:319–337, 1978.

[10] B. Courcelle. Fundamental properties of infinite trees. *Theor. Comput. Sci.*, 25:95–169, 1983.

[11] W. Damm. Higher type program schemes and their tree languages. In H. Tzschach et al., eds., *Proc. of 3rd GI Conf. on Theoretical Computer Science (Darmstadt, March 1977)*, v. 48 of *Lect. Notes in Comput. Sci.*, pp. 51–72. Springer, 1977.

[12] W. Damm. The IO and OI hierarchies. *Theor. Comput. Sci.*, 20:95–206, 1982.

[13] J. Gallier. *n*-rational algebras I: basic properties and free algebras. *SIAM J. on Comput.*, 13:750–775, 1984.

[14] J. A. Goguen, J. W. Thatcher, E. G. Wagner and J. B. Wright. Initial algebraic semantics and continuous algebras. *J. of ACM*, 24:68–95, 1977.

[15] I. Guessarian. *Algebraic Semantics*, v. 99 of *Lect. Notes in Comput. Sci.* Springer, 1981.

[16] S. Heilbrunner. An algorithm for the solution of fixed-point equations for infinite words. *Theor. Inform. and Appl.*, 14:131–141, 1980.

[17] J. B. Rosenstein. *Linear Orderings*. Academic Press, 1982.

[18] W. Thomas. On frontiers of regular trees. *Theor. Inform. and Appl.*, 20:371–381, 1986.

[19] M. Wand. Fixed point constructions in order-enriched categories. *Theor. Comput. Sci.*, 8:13–30, 1979.