An Easy Completeness Proof for the Modal μ-Calculus on Finite Trees

Balder ten Cate*
INRIA Saclay - Île-de-France and ENS Cachan
61 avenue du President Wilson, F-94235 Cachan Cedex, France
balder.tencate@gmail.com
Gaëlle Fontaine†
Institute for Logic, Language and Computation, University of Amsterdam
P.O. Box 94242, NL-1090 GE Amsterdam, The Netherlands
gaelle.fontaine@uva.nl

The μ-calculus is an extension of modal logic with a fixpoint operator. In 1983, Dexter Kozen suggested an axiomatization (see, e.g., [4]). It took more than ten years to prove completeness. This proof is due to Igor Wałukiewicz [7] and is quite involved. We propose here a simpler proof in a particular case. More precisely, we prove the completeness of the Kozen axiomatization $\mathbf{K}^\mu$ extended with the axiom $\mu x.\Box x$ with respect to the class of finite tree models.

Our argument basically consists of three steps. The first step consist of defining a notion of rank which plays the same role as the modal depth for modal formulas. One of the main properties of the rank is the following. In order to know whether a formula φ of rank n is true at a node w, it is enough to know which proposition letters are true at w and which formulas of rank at most n are true at the successor nodes of w. Another key property of the rank is that there are only finitely many formulas of a given rank (up to logical equivalence).

The second step is to prove completeness of the μ-calculus with respect to generalized models, which are basically Kripke models augmented with a set of admissible subsets, in the style of Henkin semantics for second order logic. We do this by a standard canonical model construction.

The last step is inspired by a work of Kees Doets (see, e.g., [1]). Let us call a node in a generalized model n-good if there is a node in a finite tree model which satisfies exactly the same formulas of rank at most n. Using an induction principle, we show that every node in a generalized model satisfying $\mu x.\Box x$ is n-good. It is here that we use the main property of the rank. Finally, putting this together with the completeness for generalized models, we obtain completeness for the class of finite tree models.

This argument can also be applied to some extensions of the logic $\mathbf{K}^\mu + \mu x.\Box x$. More precisely, we show that when we add finitely many shallow axioms (as defined in [6]), we obtain a complete axiomatization for the corresponding class of finite trees. We also mention that we can adapt our proof to show completeness for the graded μ-calculus extended with the axiom $\mu x.\Box x$.

The paper is organized as follows. In section 1, we recall what is the Kozen axiomatization for the μ-calculus $\mathbf{K}^\mu$ and what is the intended semantics. In section 2, we define the notion of rank for a formula. In section 3, we give a definition for the generalized models and we show completeness of $\mathbf{K}^\mu$ with respect to the class of generalized models. In section 4, we use Kees Doets' argument to obtain completeness of $\mathbf{K}^\mu + \mu x.\Box x$ with respect to the class of finite tree models. In the last two sections, we give some examples of extensions of $\mathbf{K}^\mu + \mu x.\Box x$ to which we can apply our method in order to prove completeness.

*supported by the Netherlands Organization for Scientific Research (NWO), under grant 639.021.508 and by ERC Advanced Grant Webdam on Foundation of Web data management
†supported by VICI grant 639.073.501 of the NWO

1 Syntax, Semantics and Axiomatization

We introduce the language and the Kripke semantics for the μ-calculus. We also recall the axiomatization given by Dexter Kozen.

Definition 1.1. The μ-formulas over a set *Prop* of proposition letters are given by

$$\varphi ::= \top \mid p \mid x \mid \varphi \vee \varphi \mid \neg\varphi \mid \Diamond\varphi \mid \mu x.\varphi,$$

where p ranges over the set *Prop* and x ranges over the set *Var* of variables. In $\mu x.\varphi$, we require that the variable x appears only under an even number of negations in φ. We will assume that *Var* is infinite.

As usual, we let $\phi \wedge \psi$, $\Box\varphi$ and $\nu x.\varphi$ be abbreviations for $\neg(\neg\varphi \vee \neg\psi)$, $\neg\Diamond\neg\varphi$ and $\neg\mu x.\neg[\neg x/x]$.

The notions of *subformula*, *bound variable*, *free variable* and *substitution* are defined in the usual way. If φ and ψ are μ-formulas and if p is a proposition letter, we denote by $\varphi[\psi/p]$ the formula obtained by replacing in φ each occurrence of p by ψ. Similarly, if x is a variable, we define $\varphi[\psi/x]$.

A *μ-sentence* is a formula in which all the variables are bound.

Definition 1.2. A *Kripke frame* is a pair (W,R), where W is a set and R a binary relation on W. A *Kripke model* is a triple (W,R,V) where (W,R) is a Kripke frame and $V : Prop \to \mathscr{P}(W)$ a valuation. If (w,v) belongs to R, we say that w is a *predecessor* of v and v is a *successor* of w.

Given a formula φ, a Kripke model $\mathscr{M} = (W,R,V)$ and an assignment $\tau : Var \to \mathscr{P}(W)$, we define a subset $[\![\varphi]\!]_{\mathscr{M},\tau}$ that is interpreted as the set of points at which φ is true. The subset is defined by induction in the usual way. We only recall that

$$[\![\mu x.\varphi]\!]_{\mathscr{M},\tau} = \bigcap\{U \subseteq W : [\![\varphi]\!]_{\mathscr{M},\tau[x:=U]} \subseteq U\},$$

where $\tau[x := U]$ is the assignment τ' such that $\tau'(x) = U$ and $\tau'(y) = \tau(y)$, for all $y \neq x$. Observe that the set $[\![\mu x.\varphi]\!]_{\mathscr{M},\tau}$ is the least fixpoint of the map $\varphi_x : \mathscr{P}(W) \to \mathscr{P}(W)$ defined by $\varphi_x(U) := [\![\varphi]\!]_{\mathscr{M},\tau[x:=U]}$, for all $U \subseteq W$.

If $w \in [\![\varphi]\!]_{\mathscr{M},\tau}$, we write $\mathscr{M}, w \Vdash_\tau \varphi$ and we say that φ is *true* at w under the assignment τ. If φ is a sentence, we simply write $\mathscr{M}, w \Vdash \varphi$.

A formula φ is *true* in $\mathscr{M}$ under an assignment τ if for all $w \in W$, we have $\mathscr{M}, w \Vdash_\tau \varphi$. In this case, we write $\mathscr{M} \Vdash_\tau \varphi$. A set Φ of formulas is *true* in a model $\mathscr{M}$ under an assignment τ, notation: $\mathscr{M} \Vdash_\tau \Phi$, if for all φ in Φ, φ is true in $\mathscr{M}$ under τ.

Finally, if (W,R) is a Kripke frame and for all valuations V and all assignments τ, φ is true in (W,R,V) under the assignment τ, we say that φ is *valid* in (W,R) and we write $(W,R) \Vdash \varphi$.

Definition 1.3. The axiomatization of the Kozen system $\mathbf{K}^\mu$ consists of the following axioms and rules

propositional tautologies,	
If $\vdash \varphi \to \psi$ and $\vdash \varphi$, then $\vdash \psi$	(Modus ponens),
If $\vdash \varphi$, then $\vdash \varphi[p/\psi]$	(Substitution),
$\vdash \Box(p \to q) \to (\Box p \to \Box q)$	(K-axiom),
If $\vdash \varphi$, then $\vdash \Box\varphi$	(Necessitation),
$\vdash \varphi[x/\mu x.\varphi] \to \mu x.\varphi$	(Fixpoint axiom),
If $\vdash \varphi[x/\psi] \to \psi$, then $\vdash \mu x.\varphi \to \psi$	(Fixpoint rule),

where x is not a bound variable of φ and no free variable of ψ is bound in φ.

Definition 1.4. If Φ is a set of μ-formulas, we write $\mathbf{K}^\mu + \Phi$ for the smallest set of formulas which contains both $\mathbf{K}^\mu$ and Φ and is closed for the Modus Ponens, Substitution, Necessitation and Fixpoint rules.

Definition 1.5. Let (W,R) be a Kripke frame. A point r in W is a *root* if for all w in W, there is a sequence $w_0,\dots,w_n$ such that $w_0 = r$, $w_n = w$ and (w_i,w_{i+1}) belongs to R, for all $i \in \{0,\dots,n-1\}$.

The frame (W,R) is a *tree* if it has a root, every point distinct from the root has a unique predecessor and there is no sequence $w_0,\dots,w_{n+1}$ in W such that $w_{n+1} = w_0$ and (w_i,w_{i+1}) belongs to R, for all $i \in \{0,\dots,n\}$. The frame (W,R) is a *finite tree* if it is a tree and W is finite.

Finally, a *finite tree Kripke model* is a Kripke model (W,R,V) such that (W,R) is a finite tree.

Fact 1.6. *Let $\mathscr{M} = (W,R,V)$ be a Kripke model. The formula $\mu x.\Box x$ is true at a point w in $\mathscr{M}$ iff there is no infinite sequence $w_0,w_1\dots$ in W such that $w_0 = w$ and (w_i,w_{i+1}) belongs to R, for all $i \in \mathbb{N}$.*

In particular, the formula $\mu x.\Box x$ is true in $\mathscr{M}$ iff there is no infinite sequence $w_0,w_1,\dots$ such that (w_i,w_{i+1}) belongs to R, for all $i \in \mathbb{N}$. That is, iff $\mathscr{M}$ is conversely well-founded.

We prove the completeness of the logic $\mathbf{K}^\mu + \mu x.\Box x$ with respect to the class of finite tree Kripke models. That is, a formula φ is provable in $\mathbf{K}^\mu + \mu x.\Box x$ iff it is valid in any finite tree Kripke model. Note that this result can be easily derived from the completeness result proved by Igor Wałukiewicz in [7].

2 Rank of a Formula

The goal of this section is to come up with a definition of rank that would be the analogue of the depth of a modal formula. For modal logic, it is not hard to see that the truth of an arbitrary formula φ at some world w only depends of the truth of the proposition letters at w and of the truth of formulas ψ at the successors of w, where the depth of ψ is at most the depth of φ. In our proof, we will need something similar for the μ-calculus.

The most natural idea would be to look at the nesting depth of modal and fixpoint operators. However, this definition does not have the required properties. The notion of rank that we develop in this section is in fact related to the closure of a formula, which has been introduced by Dexter Kozen in [4].

Definition 2.1. The *closure* $Cl(\varphi)$ of a formula φ is the smallest set of formulas such that

$$\begin{aligned}
&\varphi \in Cl(\varphi),\\
&\text{if } \Diamond\psi \in Cl(\varphi), \text{ then } \psi \in Cl(\varphi),\\
&\text{if } \neg\psi \in Cl(\varphi), \text{ then } \psi \in Cl(\varphi),\\
&\text{if } \mu x.\psi \in Cl(\varphi), \text{ then } \psi[x/\mu x.\psi] \in Cl(\varphi).\\
&\text{if } \psi \vee \chi \in Cl(\varphi), \text{ then both } \psi,\chi \in Cl(\varphi),
\end{aligned}$$

It is also proved in [4] that the closure $Cl(\varphi)$ of a formula φ is finite. In order to define the rank, we also need to recall the notion of the depth of a formula.

Definition 2.2. The *depth* $d(\varphi)$ of a formula φ is defined by induction as follows

$$\begin{aligned}
&d(\top) = d(p) = d(x) = 0,\\
&d(\varphi \vee \psi) = max\{d(\varphi), d(\psi)\},\\
&d(\neg\varphi) = d(\varphi),\\
&d(\Diamond\varphi) = d(\mu x.\varphi) = d(\varphi) + 1.
\end{aligned}$$

Definition 2.3. The *rank* of a formula φ is defined as follows

$$rank(\varphi) = max\{d(\psi) \mid \psi \in Cl(\varphi)\}.$$

Remark that since $Cl(\varphi)$ is finite, $rank(\varphi)$ is always a natural number. All we will use later are the following properties of the rank.

Proposition 2.4. *If the set Prop of proposition letters is finite, then for all natural numbers k, there are only finitely many sentences of rank k (up to logical equivalence).*

Proof. Fix a natural number k. Note first that if $rank(\varphi) = k$, then in particular, $d(\varphi) \leq k$. Hence, it is enough to show that there only finitely many sentences of depth below k (up to logical equivalence). If $d(\varphi) \leq k$, we may assume that the only variables occurring in φ are some $x_1, \dots, x_k$. It is routine to prove by induction on l that there are finitely many formulas of depth l with variables $x_1, \dots, x_k$. □

Proposition 2.5. *The rank is closed under boolean combination. That is, for any n, a boolean combination of formulas of rank at most n is a formula of rank at most n.*

Proposition 2.6. *Every formula φ is provably equivalent to a boolean combination of proposition letters and formulas of the form $\Diamond\psi$, with $rank(\psi) \leq rank(\varphi)$.*

Proof. Recall that a formula is guarded if every bound variable is in the scope of a modal operator. It can be shown that every formula is provably equivalent to a guarded formula. Therefore, let φ be a guarded formula. We define a map G by induction as follows:

$$\begin{array}{rcl} G(\top) & = & \top, \\ G(p) & = & p, \text{ if } p \text{ is a free variable of } \varphi, \\ G(\neg\psi) & = & \neg G(\psi), \\ G(\psi \vee \psi') & = & G(\psi) \vee G(\psi'), \\ G(\Diamond\psi) & = & \Diamond\psi, \\ G(\mu x.\psi) & = & G(\psi[x/\mu x.\psi]). \end{array}$$

Note that G is not defined for a bound variable x of φ. Using the fact that φ is guarded, one can show that the computation of $G(\varphi)$ is well-defined and does terminate. It is not hard to see that $G(\varphi)$ is equivalent to φ. Remark now that if ψ belongs to $Cl(\varphi)$, then $Cl(\psi)$ is a subset of $Cl(\varphi)$. It follows that $G(\varphi)$ is a boolean combination of proposition letters and formulas of the form $\Diamond\psi$, with $rank(\psi) \leq rank(\varphi)$. □

3 Completeness for Generalized Models

We introduce generalized models which are the analogue for the μ-calculus of the general models for second order logic. We prove completeness of $\mathbf{K}^\mu$ with respect to the class of generalized models.

Definition 3.1. Consider a quadruple $\mathscr{M} = (W, R, V, \mathbb{A})$ where (W, R) is a Kripke frame, $\mathbb{A}$ is a subset of $\mathscr{P}(W)$ and $V : Prop \to \mathbb{A}$ a valuation. A set which belongs to $\mathbb{A}$ is called *admissible*.

We define the truth of a formula φ under an assignment $\tau : Var \to \mathbb{A}$ by induction. Remark that all the clauses are the same as usual, except the one defining the truth of $\mu x.\varphi$. Normally, we define the set $[\![\mu x.\varphi]\!]_{\mathscr{M},\tau}$ as the least pre-fixpoint of the map φ_x (see Definition 1.2). But here, we define it as the intersection of all the pre-fixpoints of φ_x, that are admissible.

$$\begin{array}{l} [\![\top]\!]_{\mathscr{M},\tau} = W, \\ [\![p]\!]_{\mathscr{M},\tau} = V(p), \\ [\![x]\!]_{\mathscr{M},\tau} = \tau(x), \\ [\![\neg\varphi]\!]_{\mathscr{M},\tau} = W \backslash [\![\varphi]\!]_{\mathscr{M},\tau}, \\ [\![\varphi \vee \psi]\!]_{\mathscr{M},\tau} = [\![\varphi]\!]_{\mathscr{M},\tau} \cup [\![\psi]\!]_{\mathscr{M},\tau}, \\ [\![\Diamond\varphi]\!]_{\mathscr{M},\tau} = \{w \in W : \exists v \in W \text{ s.t. } wRv \text{ and } v \in [\![\varphi]\!]_{\mathscr{M},\tau}\}, \\ [\![\mu x.\varphi]\!]_{\mathscr{M},\tau} = \bigcap\{U \in \mathbb{A} : [\![\varphi]\!]_{\mathscr{M},\tau[x:=U]} \subseteq U\}, \end{array}$$

where $\tau[x := U]$ is the assignment τ' such that $\tau'(x) = U$ and $\tau(y) = \tau(y)$, for all $y \neq x$. If $w \in [\![\varphi]\!]_{\mathscr{M},\tau}$, we write $\mathscr{M}, w \Vdash_\tau \varphi$ and we say that φ is *true* at w under the assignment τ. If φ is a sentence, we simply write $\mathscr{M}, w \Vdash \varphi$. A formula φ is *true* in $\mathscr{M}$ under an assignment τ if for all $w \in W$, we have $\mathscr{M}, w \Vdash_\tau \varphi$. In this case, we write $\mathscr{M} \Vdash_\tau \varphi$.

The quadruple $\mathscr{M} = (W, R, V, \mathbb{A})$ is a *generalized model* if for all formulas φ and all assignments $\tau : Var \to \mathbb{A}$, the set $[\![\varphi]\!]_{\mathscr{M},\tau}$ belongs to $\mathbb{A}$. A triple $\mathscr{F} = (W, R, \mathscr{A})$ is a *generalized frame* if for every valuation $V : Prop \to \mathbb{A}$, the quadruple $(W, R, V, \mathbb{A})$ is a generalized model.

If $\mathscr{F} = (W, R, \mathscr{A})$ is a generalized frame, we call (W, R) the *underlying Kripke frame* of $\mathscr{F}$. A formula φ is *valid* in a generalized frame $\mathscr{F} = (W, R, \mathscr{A})$, notation: $\mathscr{F} \Vdash \varphi$, if for all valuations $V : Prop \to \mathbb{A}$ and all assignments $\tau : Var \to \mathbb{A}$, the formula φ is true in $(W, R, V, \mathbb{A})$ under the assignment τ.

Remark that any Kripke model $M = (W, R, V)$ can be seen as the generalized model $M' = (W, R, V, \mathscr{P}(W))$. It follows easily from our definition that for all formulas φ and all points $w \in W$,

$$M, w \Vdash \varphi \quad \text{iff} \quad M', w \Vdash \varphi.$$

Theorem 3.2. $\mathbf{K}^\mu$ *is complete with respect to the class of generalized models. That is, for any formula* φ, $\vdash_{\mathbf{K}^\mu} \varphi$ *iff for any generalized model* $\mathscr{M}$, $\mathscr{M} \Vdash \varphi$.

Proof. The argument is similar to the modal case and uses a variant of the standard canonical model construction (see, e.g.g, [5]). □

4 Completeness for Finite Tree Models

In the style of Kees Doets [1], we prove completeness of $\mathbf{K}^\mu + \mu x.\Box x$ with respect to the class of finite tree Kripke models. The argument is as follows. First, we say that a point w in a generalized model is n-good if there is a point v in a finite tree Kripke model such that no formula of rank at most n can distinguish w from v. Next, we show that "being n-good" is a property that can be expressed by a formula γ_n of rank at most n. Afterwards, we prove that each point (in a generalized model) satisfying $\mu x.\Box x$, is n-good. Finally, using completeness for generalized models, we obtain completeness of $\mathbf{K}^\mu + \mu x.\Box x$ with respect to the class of finite tree Kripke models.

In this section, we will assume that the set *Prop* of proposition letters is finite. Often we write "finite tree" instead of "finite tree Kripke model".

Definition 4.1. Fix a natural number n. Let $\mathscr{M}$ and $\mathscr{M}'$ be two generalized models. A world $w \in \mathscr{M}$ is *rank n-indistinguishable* to a world $w' \in \mathscr{M}'$ if for all formulas φ of rank at most n, we have

$$\mathscr{M}, w \Vdash \varphi \quad \text{iff} \quad \mathscr{M}', w' \Vdash \varphi.$$

In case this happens, we write $(\mathscr{M}, w) \sim_n (\mathscr{M}', w')$. Finally, we say that $w \in \mathscr{M}$ is *n-good* if there exists a finite tree $\mathscr{N}$ and some $v \in \mathscr{N}$ such that $(\mathscr{M}, w) \sim_n (\mathscr{N}, v)$.

Definition 4.2. Let n be a natural number and let Φ_n be the set of formulas of rank at most n. For any generalized model $\mathscr{M}$ and any $w \in \mathscr{M}$, we define the *n-type* $\theta_n(w)$ as the set of formulas in Φ_n which are true at w.

Remark that by Proposition 2.4, Φ_n is finite (up to logical equivalence) and in particular, there are only finitely many distinct n-types.

Lemma 4.3. *Let n be a natural number. There exists a formula* γ_n *of rank n such that for any generalized model* $\mathscr{M}$ *and any* $w \in \mathscr{M}$*, we have*

$$\mathscr{M}, w \Vdash \gamma_n \quad \textit{iff} \quad (\mathscr{M}, w) \textit{ is n-good}.$$

Proof. Let n be a natural number and let γ_n be the formula defined by

$$\gamma_n = \bigvee\{\bigwedge \theta_n(w) \mid w \text{ is } n\text{-good}\},$$

where w a point in a generalized model $\mathscr{M}$ and $\bigwedge \theta_n(w)$ is shorthand for $\bigwedge\{\varphi : \varphi \in \theta_n(w)\}$. Note that since there are only finitely many distinct n-types, the formula γ_n is well-defined. Moreover, from Proposition 2.5, it follows that the rank of γ_n is n.

It remains to check that γ_n has the required properties. It is immediate to see that if a point w in a generalized model is n-good, then γ_n is true at w. For the other direction, assume that γ_n is true at a point w in a generalized model $\mathscr{M}$. Therefore, there is a point w' in a generalized model $\mathscr{M}'$ such that w' is n-good and $\theta_n(w')$ is true at w. Since w' is n-good, there is a point v in a model $\mathscr{N}$ such that w' and v are rank n-indistinguishable. Using the fact that w and w' have the same n-type, we obtain that w and v are also rank n-indistinguishable. That is, w is n-good. □

Lemma 4.4. *For all natural numbers n, $\vdash_{\mathbf{K}^\mu} \Box\gamma_n \to \gamma_n$.*

Proof. Let n be a natural number. By Theorem 3.2, it is enough to show that the formula $\Box\gamma_n \to \gamma_n$ is valid in all generalized models. Let $\mathscr{M}$ be a generalized model and let $w \in \mathscr{M}$. We have to show $\mathscr{M}, w \Vdash \Box\gamma_n \to \gamma_n$. So suppose $\mathscr{M}, w \Vdash \Box\gamma_n$. If w is a reflexive point, we immediately obtain $\mathscr{M}, w \Vdash \gamma_n$ and this finishes the proof. Assume now that w is irreflexive. We have to prove that $(\mathscr{M}, w)$ is n-good. That is, we have to find a finite tree $\mathscr{N}$ and some $v \in \mathscr{M}$ such that $(\mathscr{M}, w) \sim_n (\mathscr{N}, v)$.

Now for any successor u of w, we have $\mathscr{M}, u \Vdash \gamma_n$. Therefore, $(\mathscr{M}, u)$ is n-good and there exists a finite tree $\mathscr{M}_u = (W_u, R_u, V_u)$ and some $w_u \in W_u$ such that $(\mathscr{M}, u) \sim_n (\mathscr{M}_u, w_u)$. Without loss of generality, we may assume that w_u is the root of $\mathscr{M}_u$.

The idea is now to look at the disjoint union of these models and to add a root v (that would be rank n-indistinguishable from w). However, this new model might not be a finite tree (w might have infinitely many successors). The solution is to restrict ourselves to finitely many successors of w. More precisely, for each n-type, we pick at most one successor of w.

So let U be a set of successors of w such that for any successor u of w, there is exactly one point u' of U satisfying $\theta_n(u) = \theta_n(u')$. Remark that since there are only finitely many distinct n-types, U is finite. Let $\mathscr{N} = (W, R, V)$ be the model defined by

$$\begin{aligned} W &= \{v\} \cup \biguplus\{W_u : u \in U\}, \\ R &= \{(v, w_u) : u \in U\} \cup \bigcup\{R_u : u \in U\}, \\ V(p) &= \begin{cases} \{v\} \cup \bigcup\{V_u(p) : u \in U\} & \text{if } \mathscr{M}, w \Vdash p, \\ \bigcup\{V_u(p) : u \in U\} & \text{otherwise,} \end{cases} \end{aligned}$$

for all proposition letters p. Since U is finite, $\mathscr{N}$ is a finite tree. Thus, it is enough to check that for any formula φ of rank at most n, we have

$$\mathscr{M}, w \Vdash \varphi \quad \text{iff} \quad \mathscr{N}, v \Vdash \varphi.$$

By Proposition 2.6, φ is provably equivalent to a boolean combination of proposition letters and formulas of the form $\Diamond\psi$, where $rank(\psi)$ is at most n. Thus, it is enough to show that w and v satisfy exactly the same proposition letters and the same formulas $\Diamond\psi$ with $rank(\psi) \leq n$.

By definition of V, it is immediate that w and v satisfy the same proposition letters. Now let ψ be a formula of rank at most n. We have to show that

$$\mathscr{M}, w \Vdash \Diamond\psi \quad \text{iff} \quad \mathscr{N}, v \Vdash \Diamond\psi.$$

For the direction from left to right, suppose that $\mathscr{M},w \Vdash \Diamond\psi$. Thus, there exists a successor u of w such that $\mathscr{M},u \Vdash \psi$. By definition of U, there is $u' \in U$ such that $(\mathscr{M},u) \sim_n (\mathscr{M},u')$. Thus, $(\mathscr{M},u) \sim_n (\mathscr{M}_{u'},w_{u'})$ and in particular, $\mathscr{M}_{u'},w_{u'} \Vdash \psi$. By definition of R, it follows that $\mathscr{N},v \Vdash \Diamond\psi$. The direction from right to left is similar. □

Proposition 4.5. *For all natural numbers n,* $\vdash_{\mathbf{K}^\mu} \mu x.\Box x \to \gamma_n$.

Proof. By Lemma 4.4, we know that $\Box\gamma_n \to \gamma_n$ is provable in $\mathbf{K}^\mu$. By the Fixpoint rule, we obtain that $\mu x.\Box x \to \gamma_n$ is provable in $\mathbf{K}^\mu$. □

Theorem 4.6. $\mathbf{K}^\mu + \mu x.\Box x$ *is complete with respect to the class of finite tree Kripke models.*

Proof. For any finite tree $\mathscr{M}$, we have $\mathscr{M} \Vdash \mathbf{K}^\mu$ and $\mathscr{M} \Vdash \mu x.\Box x$. Thus, it is sufficient to show that if φ is not provable in $\mathbf{K}^\mu + \mu x.\Box x$, there exists a finite tree N such that $N \nVdash \varphi$. Let φ be such a formula. In particular, $\nvdash_{\mathbf{K}^\mu} \mu x.\Box x \to \varphi$. By Theorem 3.2, we have $\mathscr{M},w \nVdash \mu x.\Box x \to \varphi$, for some generalized model $\mathscr{M}$ and some $w \in \mathscr{M}$.

Let n be the rank of φ. By Theorem 3.2 and Proposition 4.5, we get that $\mathscr{M},w \Vdash \mu x.\Box x \to \gamma_n$. Since $\mathscr{M},w \Vdash \mu x.\Box x$, it follows that $\mathscr{M},w \Vdash \gamma_n$. Therefore, there exists a finite tree $\mathscr{N}$ and some $v \in \mathscr{N}$ such that $(\mathscr{M},w) \sim_n (\mathscr{N},v)$. Since $\mathscr{M},w \nVdash \varphi$, we have $\mathscr{N},v \nVdash \varphi$. □

As mentioned before, this result also follows from the completeness of $\mathbf{K}^\mu$ showed by Igor Wałukiewicz in [7]. We briefly explain how to derive Theorem 4.6 from the completeness of $\mathbf{K}^\mu$. Recall that in [7], Igor Walukiewicz showed that a sentence φ is provable in $\mathbf{K}^\mu$ iff it is valid in all trees.

Suppose that a sentence φ is not provable in $\mathbf{K}^\mu + \mu x.\Box x$. In particular, the formula $\mu x.\Box x \to \varphi$ is not provable in $\mathbf{K}^\mu$. It follows from the completeness of $\mathbf{K}^\mu$ that there is a model $\mathscr{M} = (W,R.V)$ and a point w in W such that (W,R) is a tree and $\mu x.\Box x \to \varphi$ is not true at w. We may assume that w is the root of (W,R).

Since $\mu x.\Box x$ is true at w and since w is the root, it follows from Fact 1.6 that the tree (W,R) is conversely well-founded. Let n be the rank of φ. Now, if a point v in W has more than one successor of a given n-type θ, we can pick one successor of n-type θ and delete all the other successors of n-type θ. This would not modify the fact that φ is not true at w. By doing this operation inductively and using the fact that (W,R) is well-founded, we can prove that the tree (W,R) may be assumed to be finite. Therefore, there is a finite tree (W,R) in which φ is not valid.

5 Adding Shallow Axioms to $\mathbf{K}^\mu + \mu x.\Box x$

By slightly modifying our method, it is also possible to prove that the logic obtained by adding the axiom $\Diamond p \to \Box p$ to $\mathbf{K}^\mu + \mu x.\Box x$ is complete with respect to the class of finite strings (recall that a string is a tree such that every point has at most one successor).

We do not provide the details of the proof but it consists in two parts. First, we show that if we construct a canonical generalized model for this logic, then the underlying Kripke frame satisfies the axiom $\Diamond p \to \Box p$. Second, we modify the definition of being n-good by requiring in Definition 4.1 that the model $\mathscr{N}$ is a finite string. Then we prove that the lemmas 4.3 and 4.4 still holds for this new definition.

Theorem 5.1. *The logic* $\mathbf{K}^\mu + \mu x.\Box x + (\Diamond p \to \Box p)$ *is complete with respect to the class of finite strings.*

We remark that this theorem follows from a result by Roope Kaivola (see, e.g., [3]). But the proof proposed here is simpler.

More generally, we can also show that when we extend the logic $\mathbf{K}^\mu + \mu x.\Box x$ with axioms that are shallow (defined below), we obtain complete axiomatizations for the corresponding class of finite trees.

Definition 5.2 ([6])**.** A formula is *shallow* if no occurrence of a proposition letter is in the scope of a fixpoint operator and each occurrence of a proposition letter is in the scope of at most one modality. In other words, the shallow formulas is the language defined by

$$\varphi \ ::= \ \psi \mid \Diamond\psi \mid \varphi \vee \varphi \mid \neg\varphi,$$

where ψ is either a formula without any proposition letter or a propositional formula (that is a formula of the μ-calculus that does not contain neither $\Diamond$ nor μ).

Observe that the formula $\Diamond p \to \Box p$ is a shallow formula. Other examples are formulas expressing that each point has at most two successors ($\Diamond p \wedge \Diamond(q \vee \neg p) \to \Box(p \vee q)$), or that each point has at most one blind successor ($\Diamond(p \wedge \Box\bot) \wedge \Box(\Box\bot \to p)$).

Recall that a formula φ defines a class $\mathscr{C}$ of finite trees if $\mathscr{C}$ is exactly the class of trees which make φ valid.

Theorem 5.3. *Let φ be a shallow formula. Then the logic $\mathbf{K}^{\mu} + \mu x.\Box x + \varphi$ is complete with respect to the class of finite trees defined by φ.*

The structure of the proof is similar to the one of the proof of Theorem 5.1. Here, in order to show that the underlying frame of the canonical generalized model satisfies φ, we use the fact that the shallow formulas are canonical, which was proved in [6].

6 Graded μ-Calculus

Finally, we would like to mention that we can also use the same method to show that we can obtain a complete axiomatization for the graded μ-calculus together with the axiom $\mu x.\Box x$. In [2], Maurizio Fattorosi-Barnaba and Claudio Cerrato gave an axiomatization of graded modal logic and show that this axiomatization was complete with respect to the class of frames. If we add the Fixpoint axiom, the Fixpoint rule and the axiom $\mu x.\Box x$ to their axiomatization, we obtain a logic that is complete with respect to the class of finite trees.

The only part of the proof which requires some extra work is when we want to show a result similar to Theorem 3.2. Indeed, the canonical construction for graded modal logic is already not very easy. In fact, in order to show completeness for graded μ-calculus with respect to the class of generalized frames, we use directly the completeness result by Maurizio Fattorosi-Barnaba and Claudio Cerrato, instead of going trough the canonical model construction. This is done by translating each μ-formula into a modal formula, but over a larger set of proposition letters. We do not give the details, by lack of space.

Acknowledgements

We would like to thank Alexandru Baltag and Yde Venema for their comments on earlier drafts.

References

[1] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*, v. 53 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge Univ. Press, 2001.

[2] B. ten Cate. *Model Theory for Extended Modal Languages*. PhD thesis, University of Amsterdam, 2005. ILLC Dissertation Series DS-2005-01.

[3] K. Doets. Monadic Π^1_1-theories of Π^1_1 properties. *Notre Dame J. of Formal Log.*, 30(2):224–240, 1989.

[4] M. Fattorosi-Barnaba and C. Cerrato. Graded modalities III: the completeness and compactness of $S4^0$. *Studia Logica*, 47(2):99–110, 1988.

[5] R. Kaivola. *Using Automata to Characterise Fixed Point Temporal Logics.* PhD thesis, University of Edinburgh, 1997.

[6] D. Kozen. Results on the propositional μ-calculus. *Theor. Comput. Sci.*, 27:333–354, 1983.

[7] I. Wałukiewicz. A note on the completeness of Kozen's axiomatization of the propositional μ-calculus. *Bull. of Symb. Log.*, 2(3):349–366, 1996.