A Note on the Relation between Inflationary Fixpoints and Least Fixpoints of Higher-Order

Stephan Kreutzer
Oxford University Computing Laboratory
Wolfson Building, Parks Road,
Oxford OX1 3QD, United Kingdom
kreutzer@comlab.ox.ac.uk

Martin Lange
Institut für Informatik,
Ludwig-Maximilians-Universität München
Oettingenstr. 67, D-80538 München, Germany
martin.lange@ifi.lmu.de

Abstract

Least fixpoints of monotone functions are an important concept in computer science which can be generalised to inflationary fixpoints of arbitrary functions. This raises questions after the expressive power of these two concepts, in particular whether the latter can be expressed as the former in certain circumstances. We show that the inflationary fixpoint of an arbitrary function on a lattice of finite height can be expressed as the least fixpoint of a monotone function on an associated function lattice.

1 Introduction

Possibly the most important type of fixpoints in computer science are least fixpoints of monotone functions, with countless concepts and definitions being based on this principle, e.g. abstract data types, formal languages, semantics of programming constructs, static analysis algorithms, logical operators.

In mathematical logic, fixpoint inductions over definable functions on arbitrary structures have first been studied in generalised recursion theory (see [10]), following earlier work in recursion theory on inductive definitions in arithmetic. If $\varphi(X,x)$ is a first-order formula with a free first-order variable x and a free second-order variable X, which we call the fixpoint variable of φ, then φ defines on any structure $\mathfrak{A}$ with universe A a function f_φ on the powerset lattice on A with $f_\varphi(B) := \{a \in A : (A,a) \models_{[X \mapsto B]} \varphi(X,x)\}$. Of particular interest are formulas defining a monotone function as by Knaster and Tarski's theorem (see Section 2) every monotone function has a unique least fixpoint which can also be obtained by an explicit induction process. As monotonicity of a function is in general undecidable, first-order formulas that are positive in X (and therefore monotone) are usually considered only.

A similar but seemingly more general concept of fixpoints are provided by *inflationary* fixpoints, which exist for any function, even if they are non-monotone (see Section 2 for details). In the context of logic, inflationary fixpoints of definable functions have first been studied in the 1970s (see e.g. [11, 1]) and it has been realised that not every inflationary fixpoint over an arbitrary first-order formula can also be described as a least fixpoint over a formula positive in its fixpoint variable. This naturally leads to the question of which inflationary fixpoints can equivalently be written as least fixpoints of monotone functions.

Following this early work on fixpoint inductions over definable functions, logics featuring explicit fixpoint constructs have been studied in finite model theory and in temporal logics as means to describe classes of structures or the behavior of programs for instance. The main and decisive difference to the studies in generalised recursion theory was the introduction of explicit operators to form least or inflationary fixpoint of definable functions, which allows to nest fixpoint operators and use them in the scope of negations.

Initiated by Gurevich [4], logics involving fixpoint constructs have intensively been studied in finite model theory and descriptive complexity as an elegant way to describe computational problems in logical languages (see [3] for an extensive study of fixpoint logics). Again, the most important fixpoint logics considered in this context are logics extending first-order logic by operators to form the least fixpoint of formulas positive in their fixpoint variable or operators to form the inflationary fixpoints of arbitrary formulas. It turns out that combining first-order logic with the ability to nest and complement fixpoint operators is powerful enough so that every formula of inflationary fixpoint logic is equivalent to a formula using least fixpoints of formulas positive in their fixpoint variable. This was first proved in the context of finite structures by Gurevich and Shelah [5] and then generalised by Kreutzer [7] to arbitrary structures.

In the context of modal logics, fixpoints occur most prominently in the modal μ-calculus L_μ introduced by Kozen [6]. The importance of the μ-calculus stems from its fine balance between expressive power and complexity, as it is expressive enough to encompass commonly used specification logics such as LTL, CTL and CTL*. On the other hand, it model checking problem on finite structures is in NP $\cap$ coNP and its satisfiability/validity problem is EXPTIME-complete. Besides its expressive power, the μ-calculus is still a *regular* logic because it can be embedded into monadic second-order logic (MSO). In fact, L_μ is the bisimulation-invariant fragment of MSO and hence is the most expressive regular logic invariant under bisimulation.

Being a regular logic comes with a range of restrictions, in particular the inability to count. Hence, specifications such as *a particular event occurs on all execution traces at the same time* or *every request is acknowledged* cannot be expressed in L_μ. To overcome the restriction to regular logics, extensions of the modal μ-calculus have been studied in the literature. Among those one can broadly distinguished between "first-order" fixpoint logics, i.e. logics where the fixpoint is still taken over definable functions from sets of vertices to sets of vertices but more general fixpoint constructs are allowed that least fixpoints over monotone functions, and "higher-order" fixpoint logics, where we retain monotone fixpoint inductions but allow fixpoints of operators over a function space. An example for the first approach is the *modal iteration calculus* (MIC), introduced in [2], the extension of modal logic by operators to form inflationary fixpoints of definable functions. An example of the latter is *fixpoint logic with chop* (FLC) introduced in [12], where the semantics of μ-calculus formulas is lifted from the powerset lattice of all *predicates* to the lattice of *predicate transformers* which are first-order functions from the original powerset lattice into itself. This concept has then been generalised to *higher-order fixpoint logic* (HFL), introduced in [14], which incorporates into the μ-calculus a simply typed λ-calculus used to describe predicate transformers, and functions of predicate transformers, and functions of functions of ..., etc.

For all these logics examples of non-regular properties definable in the logic have been exhibited, separating them from the modal μ-calculus. However, very little is known about the relationship between these logics. A simple complexity-theoretic argument shows that FLC cannot be embedded into MIC [8]: the expression complexity for FLC is EXPTIME-hard, i.e. there is a fixed formula s.t. model checking with this formula is already EXPTIME-hard [9]. On the other hand, MIC's data complexity is in P. Thus, if this particular FLC-formula was translatable into MIC then we would have EXPTIME $=$ P which contradicts the time hierarchy theorem. It is open whether or not MIC is translatable into FLC.

This should be seen in the more general context of the question whether monotone fixpoint of higher order can be used to express non-monotone fixpoints of first order and if this cannot be achieved in general, then under which circumstances inflationary fixpoints can be expressed as monotone fixpoints of higher order. (Note that MIC uses inflationary fixpoints of functions of type $\tau \to \tau$ while FLC uses least fixpoints of functions of type $(\tau \to \tau) \to (\tau \to \tau)$.)

The purpose of this paper is to stipulate a discussion of this problem. To initiate this we present a general result on fixpoints in complete lattices show that any inflationary fixpoint on a complete lattice of finite height can be expressed as a least fixpoint of a monotone operator on a function space associated with the lattice. As a consequence, we obtain some embeddability results for modal fixpoint logics.

2 Complete Lattices and Fixpoints

2.1 Lattices

A *partial order* is a pair $(M,\leq)$ s.t. M is a set and $\leq$ is a reflexive, anti-symmetric and transitive binary relation on M. As usual, we write $<$ for the strict relation obtained from it, i.e. $< := \leq \setminus =$.

An *upper*, resp. *lower bound* for a $N \subseteq M$ is a $y \in M$ s.t. $x \leq y$, resp. $y \leq x$, for all $x \in N$. A *maximum*, resp. *minimum*, of some $N \subseteq M$ is a $y \in N$ s.t. there is no $x \in N$ with $y < x$, resp. $x < y$. A *supremum*, resp. *infimum*, of some $N \subseteq M$ is a minimum of all upper bounds, resp. maximum of all lower bounds. As usual, we write $\bigsqcup N$, resp. $\bigsqcap N$, for the supremum, resp. infimum, of N if it exists uniquely. If $N = \{x,y\}$ we also use infix relation $x \sqcup y$, resp. $x \sqcap y$.

A *lattice* is a partial order $(M,\leq)$ s.t. for every $x,y \in M$ the supremum $x \sqcup y$ and the infimum $x \sqcap y$ exists uniquely in M. It is *complete* if $\bigsqcup N$ and $\bigsqcap N$ exist uniquely in M for every $N \subseteq M$. We define $\bot = \bigsqcap M$ and $\top = \bigsqcup M$ as the bottom and top element of a complete lattice.

Function lattices Let $\mathscr{M} = (M,\leq_{\mathscr{M}})$ and $\mathscr{N} = (N,\leq_{\mathscr{N}})$ be lattices. The space of all functions from $\mathscr{M}$ to $\mathscr{N}$ is $\mathscr{M} \to \mathscr{N} := (\{f \mid f : M \to N\},\leq)$, where

$$f \leq g \text{ iff } \forall x \in M : f(x) \leq_{\mathscr{N}} g(x)$$

Clearly, $\mathscr{M}$ need not be a lattice, not even a partial order, for the function space to be a lattice. If $\mathscr{N}$ is a lattice then so is $\mathscr{M} \to \mathscr{N}$ with

$$(f \sqcup g)(x) \;=\; f(x) \sqcup_{\mathscr{N}} g(x) \qquad (f \sqcap g)(x) \;=\; f(x) \sqcap_{\mathscr{N}} g(x)\ .$$

If $\mathscr{N}$ is complete, then so is $\mathscr{M} \to \mathscr{N}$.

2.2 Fixpoints

Let $\mathscr{M} = (M,\leq_{\mathscr{M}})$ and $\mathscr{N} = (N,\leq_{\mathscr{N}})$ be partial orders. A function $f : M \to N$ is called *monotone* if for all $x,y \in M$: if $x \leq_{\mathscr{M}} y$ then $f(x) \leq_{\mathscr{N}} f(y)$.

Let $\mathscr{M} = (M,\leq)$ be a lattice and $f : M \to M$. A *least fixpoint* of f is an element $x \in M$ s.t. $f(x) = x$ and there is no $y < x$ s.t. $f(y) = y$. Probably the most famous fixpoint theorem is Knaster-Tarski's which states unique existence of least fixpoints in case of monotone functions on complete lattices. We write μf or $\mu x.f(x)$ for *the* least fixpoint of f if it exists uniquely.

Theorem 1 ([13]). *Let $\mathscr{M} = (M,\leq)$ be a complete lattice and $f : M \to M$ monotone. Then $\mu f = \bigsqcap\{y \mid f(y) \leq y\}$.*

Another characterisation of least fixpoints of monotone functions is given by *fixpoint iteration* stating that the least fixpoint also equals the supremum of all its *approximants* $\mu^{\alpha} f$ for any ordinal α, defined as follows.

$$\mu^0 f := \bot \ , \quad \mu^{\alpha+1} f = f(\mu^{\alpha} f) \ , \quad \mu^{\kappa} f = \bigsqcup_{\alpha<\kappa} \mu^{\alpha} f$$

where κ is a limit ordinal. Then $\mu f = \bigsqcup_{\alpha} \mu^{\alpha} f$.

It is well-known and easy to show by induction that the sequence of approximants is monotonically increasing, i.e. for all ordinals α,β: if $\alpha \leq \beta$ then $\mu^{\alpha} f \leq_{\mathscr{M}} \mu^{\beta} f$. This, however, requires monotony and is not true in general for non-monotonic functions. On the other hand, the monotonous increase of the sequence is appealing for it is bound to become stable – possibly at some transfinite ordinal. Stability of course means reaching a fixpoint. If f is not monotone then one can enforce a monotonically increasing and eventually stable sequence by making it *inflationary*. The *inflationary fixpoint* of an arbitrary function $f : M \to M$ is written $\mathsf{ifp} f$ or $\mathsf{ifp} x.f(x)$ and is defined as $\bigsqcup_\alpha \mathsf{ifp}^\alpha f$ where

$$\mathsf{ifp}^0 f := \bot \ , \quad \mathsf{ifp}^{\alpha+1} f = \mathsf{ifp}^\alpha f \sqcup_{\mathscr{M}} f(\mathsf{ifp}^\alpha f) \ , \quad \mathsf{ifp}^\kappa f = \bigsqcup_{\alpha<\kappa} \mathsf{ifp}^\alpha f$$

It is not difficult to see that inflationary fixpoints are at least as expressive as least fixpoints. If f is monotone then $\mathsf{ifp} f = \mu f$. In fact, the correspondence is even stronger: $\mathsf{ifp}^\alpha f = \mu^\alpha f$ for every ordinal α. Hence, for monotone functions inflationary and least fixpoints not only coincide, they inherently are the same. This raises the question after the converse: can inflationary fixpoints be expressed in terms of least fixpoints? The next section shows that this is sometimes the case. Note that, for an arbitrary function $f : M \to M$, the function $f' : M \to M$, defined as $f'(x) = x \sqcup f(x)$ is in general not monotone and may therefore not have a (unique) least fixpoint.

In order to prove a correspondence between inflationary fixpoints and least fixpoints of higher-order in the following section we generalise the context of inflationary fixpoint iteration. Let $\mathscr{M} = (M, \leq)$ be a complete lattice, $x \in M$ and $f : M \to M$. Define $\mathsf{ifp}_x f = \bigsqcup_\alpha \mathsf{ifp}_x^\alpha f$ where

$$\mathsf{ifp}_x^0 f := x \ , \quad \mathsf{ifp}_x^{\alpha+1} f = \mathsf{ifp}_x^\alpha f \sqcup_{\mathscr{M}} f(\mathsf{ifp}_x^\alpha f) \ , \quad \mathsf{ifp}_x^\kappa f = \bigsqcup_{\alpha<\kappa} \mathsf{ifp}_x^\alpha f$$

with κ being a limit ordinal. Hence, $\mathsf{ifp}_x f$ is simply the inflationary fixpoint of f when the iteration is started in x and therefore $\mathsf{ifp} f = \mathsf{ifp}_\bot f$.

The *closure ordinal* of a function f and an element $x \in M$ is the least ordinal α s.t. $\mathsf{ifp}_x^{\alpha+1} f = \mathsf{ifp}_x^\alpha f$. It is denote $cl_x(f)$. Note that $\mathsf{ifp}_x f = \mathsf{ifp}_x^{cl_x(f)} f$. We will also write $cl(f)$ instead of $cl_\bot(f)$ where $\bot$ is the infimum of the underlying complete lattice.

3 Expressing Inflationary Fixpoints as Least Fixpoints of Higher-Order

Before we can show expressibility of inflationary fixpoints through higher-order least ones we need to prove two facts about generalised inflationary fixpoints.

Lemma 2. *Let $\mathscr{M} = (M, \leq)$ be a complete lattice, $x \in M$, and $f : M \to M$. For all ordinals $\alpha < \omega$ we have* $\mathsf{ifp}_x^{\alpha+1} f = \mathsf{ifp}_{x \sqcup f(x)}^\alpha f$.

Proof. By induction on α. The base case is $\mathsf{ifp}_x^1 = \mathsf{ifp}_x^0 f \sqcup f(\mathsf{ifp}_x^0 f) = x \sqcup f(x) = \mathsf{ifp}_{x \sqcup f(x)}^0 f$. The step case is $\mathsf{ifp}_x^{\alpha+2} f = \mathsf{ifp}_x^{\alpha+1} f \sqcup f(\mathsf{ifp}_x^{\alpha+1} f) = \mathsf{ifp}_{x \sqcup f(x)}^\alpha f \sqcup f(\mathsf{ifp}_{x \sqcup f(x)}^\alpha f) = \mathsf{ifp}_{x \sqcup f(x)}^{\alpha+1} f$. ☐

Lemma 3. *Let $\mathscr{M} = (M, \leq)$ be a complete lattice, $x \in M$, and $f : M \to M$. Then we have $x \sqcup \mathsf{ifp}_{x \sqcup f(x)} f \leq \mathsf{ifp}_x f$.*

Proof. Note that $x \leq \mathsf{ifp}_x f$. Thus, it suffices to show $\mathsf{ifp}_{x \sqcup f(x)} f \leq \mathsf{ifp}_x f$. We will separate this into two parts. First, we will show that for every ordinal $\alpha < \omega$ we have $\mathsf{ifp}_{x \sqcup f(x)}^\alpha f = \mathsf{ifp}_x^{\alpha+1} f$. This is done by induction on α. The base case is simple: $\mathsf{ifp}_{x \sqcup f(x)}^0 f = x \sqcup f(x) = \mathsf{ifp}_x^1 f$. In the step case we have

$$\mathsf{ifp}_{x \sqcup f(x)}^{\alpha+1} f \ = \ \mathsf{ifp}_{x \sqcup f(x)}^\alpha \sqcup f(\mathsf{ifp}_{x \sqcup f(x)}^\alpha f) \ = \ \mathsf{ifp}_x^{\alpha+1} f \sqcup f(\mathsf{ifp}_x^{\alpha+1} f) \ = \ \mathsf{ifp}_x^{\alpha+2} f$$

Thus, we have

$$\mathsf{ifp}_{x \sqcup f(x)}^\omega f \ = \ \bigsqcup_{\alpha<\omega} \mathsf{ifp}_{x \sqcup f(x)}^\alpha f \ = \ \bigsqcup_{\alpha<\omega} \mathsf{ifp}_x^{\alpha+1} f \ = \ \bigsqcup_{1 \leq \alpha<\omega} \mathsf{ifp}_x^\alpha f \ = \ \bigsqcup_{\alpha<\omega} \mathsf{ifp}_x^\alpha f \ = \ \mathsf{ifp}_x^\omega f \qquad (1)$$

because $\mathsf{ifp}^0_x f = x \leq \mathsf{ifp}^1_x f$.

In the second part we show that for all ordinals $\alpha \geq \omega$ we have $\mathsf{ifp}^\alpha_{x \sqcup f(x)} f = \mathsf{ifp}^\alpha_x f$. Again, this is done by induction on α, and the base case of $\alpha = \omega$ is done in Eq. (1). The case for successor odinals is similar to the first part of the proof.

$$\mathsf{ifp}^{\alpha+1}_{x \sqcup f(x)} f \;=\; \mathsf{ifp}^\alpha_{x \sqcup f(x)} \sqcup f(\mathsf{ifp}^\alpha_{x \sqcup f(x)} f) \;=\; \mathsf{ifp}^\alpha_x f \sqcup f(\mathsf{ifp}^\alpha_x f) \;=\; \mathsf{ifp}^{\alpha+1}_x f$$

using the hypothesis twice. Finally, the case of limit ordinals is easy, too.

$$\mathsf{ifp}^\kappa_{x \sqcup f(x)} f \;=\; \bigsqcup_{\alpha<\kappa} \mathsf{ifp}^\alpha_{x \sqcup f(x)} f \;=\; \bigsqcup_{\omega \leq \alpha < \kappa} \mathsf{ifp}^\alpha_{x \sqcup f(x)} f \;=\; \bigsqcup_{\omega \leq \alpha < \kappa} \mathsf{ifp}^\alpha_x f \;=\; \mathsf{ifp}^\kappa_x f$$

using the hypothesis on each approximant and the fact that $\mathsf{ifp}^\alpha_y f \leq \mathsf{ifp}^\omega_y f$ for every $\alpha < \omega$ and any y.

Thus, we have $\mathsf{ifp}_{x \sqcup f(x)} f = \mathsf{ifp}_x f$ and therefore in particular $x \sqcup \mathsf{ifp}_{x \sqcup f(x)} f \leq \mathsf{ifp}_x f$ which was to be shown. $\square$

Let $\mathscr{M} = (M, \leq_{\mathscr{M}})$ be a complete lattice and $f : M \to M$ be an arbitrary function, not necessarily monotone. Let $\mathscr{M} \to \mathscr{M} = (M \to M, \leq)$ be the complete lattice of functions from $\mathscr{M}$ to $\mathscr{M}$ with the pointwise order defined above. Define a function $F_f : (M \to M) \to (M \to M)$ as follows.

$$F_f(g) \;= \lambda x.\Big(x \sqcup g\big(x \sqcup f(x)\big)\Big)$$

Lemma 4. *Let $\mathscr{M} = (M, \leq)$ be a partial order, and $f : M \to M$ arbitrary. Then F_f is monotone w.r.t. to the partial order of the function space $\mathscr{M} \to \mathscr{M}$.*

Proof. Suppose g, g' are functions of type $M \to M$ with $g \leq g'$, i.e. $g(x) \leq_{\mathscr{M}} g'(x)$ for every $x \in M$. Then $x \sqcup (g(x \sqcup f(x))) \leq_{\mathscr{M}} x \sqcup (g'(x \sqcup f(x)))$ for every such x and therefore $F_f(g) \leq F_f(g')$. $\square$

Hence, according to the Knaster-Tarski-Theorem (Theorem 1), F_f always possesses a least fixpoint. Next we will show that this can be used to define the inflationary fixpoint of f.

Theorem 5. *Let $\mathscr{M} = (M, \leq)$ be a complete lattice with bottom element $\bot$ and $f : M \to M$ arbitrary. If $cl(f) \leq \omega$ then $\mathsf{ifp} f = (\mu F_f)(\bot)$.*

Proof. ("$\leq$") We will prove a stronger statement: for all $x \in M$ and all $\alpha \leq \omega$ we have $\mathsf{ifp}^\alpha_x f \leq (\mu F_f)(x)$. In the base case of $\alpha = 0$ we have

$$\begin{aligned}\mathsf{ifp}^0_x f \;&=\; x \;\leq\; x \sqcup \Big(\mu^0 F_f\big(x \sqcup f(x)\big)\Big) \;=\; \Big(\lambda y.y \sqcup \Big(\mu^0 F_f\big(y \sqcup f(y)\big)\Big)\Big)(x) \;=\; (\mu^1 F_f)(x) \\ &\leq\; (\mu F_f)(x)\end{aligned}$$

In the step case we have

$$\begin{aligned}\mathsf{ifp}^{\alpha+1}_x f \;&=\; \mathsf{ifp}^\alpha_{x \sqcup f(x)} f \;\leq\; x \sqcup \mathsf{ifp}^\alpha_{x \sqcup f(x)} f \;\leq\; x \sqcup (\mu F_f)\big(x \sqcup f(x)\big) \;=\; \Big(\lambda y.y \sqcup (\mu F_f)\big(y \sqcup f(y)\big)\Big)(x) \\ &=\; (\mu F_f)(x)\end{aligned}$$

according to Lemma 2 and the fact that μF_f is a fixpoint of the function F_f. Finally,

$$\mathsf{ifp}^\omega_x f \;=\; \bigsqcup_{\alpha<\omega} \mathsf{ifp}^\alpha_x f \;\leq\; \bigsqcup_{\alpha<\omega} (\mu F_f)(x) \;=\; (\mu F_f)(x)$$

using the hypothesis for every finite ordinal α.

("$\geq$") According to Lemma 3 we have $x \sqcup \mathsf{ifp}_{x \sqcup f(x)} f \leq \mathsf{ifp}_x f$ for all $x \in M$. Hence, by $\alpha\beta$-expansion we have

$$\big(\lambda y.y \sqcup (\lambda x.\mathsf{ifp}_x f)(y \sqcup f(y))\big) \;=\; (\lambda x.x \sqcup \mathsf{ifp}_{x \sqcup f(x)} f) \;\leq_{\mathscr{M} \to \mathscr{M}}\; (\lambda x.\mathsf{ifp}_x f)$$

which shows that $\lambda x.\mathsf{ifp}_x f$ is a pre-fixpoint of F_f. According to the Knaster-Tarski-Theorem (Thm. 1) we then have

$$\mu F_f \;\leq_{\mathscr{M} \to \mathscr{M}}\; \lambda x.\mathsf{ifp}_x f$$

which immediately yields $(\mu F_f)(x) \leq \mathsf{ifp}_x f$ for any $x \in M$, in particular $(\mu F_f)(\perp) \leq \mathsf{ifp} f$. $\square$

4 Conclusion and Further Work

An almost immediate consequence of Theorem 5 concerns the expressive power of temporal logics extending the modal μ-calculus: the modal iteration calculus can be embedded into the first-order fragment of higher-order fixpoint logic when interpreted over finite models only. Formulas of the former can inductively be transformed into formulas of the latter. The only difficult case is that of inflationary fixpoint quantifiers which are then handled by Theorem 5. The rest is easy because both logics extend modal logic. It remains to be seen in detail whether Theorem 5 can also explain the equi-expressiveness of first-order logic with inflationary fixpoints to first-order logic with least fixpoints on finite structures.

Clearly, the result presented here does not answer all questions about the relationship between least and inflationary fixpoints. Most importantly, it remains to be seen whether Theorem 5 can be extended to lattices of arbitrary height.

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