

Deriving derivation rules from truth tables: classically, constructively and proof reduction

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Classical and Constructive Logic

Classically, the meaning of a propositional connective is fixed by its **truth table**. This immediately implies

- consistency,
- a decision procedure,
- completeness (w.r.t. Boolean algebra's).

Constructively (following the **Brouwer-Heyting-Kolmogorov** interpretation), the meaning of a connective is fixed by explaining what a **proof** is that involves the connective.

Basically, this explains the **introduction rule(s)** for each connective, from which the elimination rules follow (Prawitz)

By analysing constructive proofs we then also get

- consistency (from proof normalization),
- a decision procedure (from the subformula property),
- completeness (w.r.t. Heyting algebra's and Kripke models).

This talk

- Derive natural deduction rules for a connective from its truth table definition.
 - Also works for constructive logic.
 - Gives natural deduction rules for a connective “in isolation”
 - Also gives (constructive) rules for connectives that haven’t been studied so far, like **if-then-else** and **nand**.
- General definition, both the constructive and the classical case.
- Relation to “standard” natural deduction rules and known connectives.
- General Kripke model for the constructive connectives. (Sound and Complete)
- Curry-Howard **proofs-as-terms** interpretation for derivations and normalization of proof-reduction
- Interpreting classical proofs as terms.

Standard form for natural deduction rules

$$\frac{\Gamma \vdash A_1 \quad \dots \quad \Gamma \vdash A_n \quad \Gamma, B_1 \vdash D \quad \dots \quad \Gamma, B_m \vdash D}{\Gamma \vdash D}$$

If the conclusion of a rule is $\Gamma \vdash D$, then the hypotheses of the rule can be of one of two forms:

- ① $\Gamma \vdash A$: instead of proving D from Γ , we now need to prove A from Γ . We call A a **Lemma**.
- ② $\Gamma, B \vdash D$: we are given extra data B to prove D from Γ . We call B a **Casus**.

We don't give the Γ explicitly (it can be retrieved):

$$\frac{\vdash A_1 \quad \dots \quad \vdash A_n \quad B_1 \vdash D \quad \dots \quad B_m \vdash D}{\vdash D}$$

Some well-known constructive rules

Rules that follow this format:

$$\frac{\vdash A \vee B \quad A \vdash D \quad B \vdash D}{\vdash D} \vee\text{-el} \qquad \frac{\vdash A \wedge B \quad A \vdash D}{\vdash D} \wedge\text{-el}$$

$$\frac{\vdash A \quad \vdash B}{\vdash A \wedge B} \wedge\text{-in}$$

Rule that does not follow this format:

$$\frac{A \vdash B}{\vdash A \rightarrow B} \rightarrow\text{-in}$$



Natural Deduction rules from truth tables

Let c be an n -ary connective c with truth table t_c .
Each row of t_c gives rise to an elimination rule or an introduction rule for c . (We write $\Phi = c(A_1, \dots, A_n)$.)

$$\frac{A_1 \quad \dots \quad A_n \mid \Phi}{p_1 \quad \dots \quad p_n \mid 0} \mapsto \frac{\vdash \Phi \dots \vdash A_j \text{ (if } p_j = 1) \dots A_i \vdash D \text{ (if } p_i = 0) \dots}{\vdash D} \text{el}$$

constructive intro

$$\frac{A_1 \quad \dots \quad A_n \mid \Phi}{q_1 \quad \dots \quad q_n \mid 1} \mapsto \frac{\dots \vdash A_j \text{ (if } q_j = 1) \dots A_i \vdash \Phi \text{ (if } q_i = 0) \dots}{\vdash \Phi} \text{in}^i$$

classical intro

$$\frac{A_1 \quad \dots \quad A_n \mid \Phi}{r_1 \quad \dots \quad r_n \mid 1} \mapsto \frac{\Phi \vdash D \dots \vdash A_j \text{ (if } r_j = 1) \dots A_i \vdash D \text{ (if } r_i = 0) \dots}{\vdash D} \text{in}$$

Examples

Constructive rules for \wedge (3 elim rules and one intro rule):

A	B	$A \wedge B$
0	0	0
0	1	0
1	0	0
1	1	1

$$\frac{\vdash A \wedge B \quad A \vdash D \quad B \vdash D}{\vdash D} \wedge\text{-el}_{00}$$

$$\frac{\vdash A \wedge B \quad A \vdash D \quad \vdash B}{\vdash D} \wedge\text{-el}_{01}$$

$$\frac{\vdash A \wedge B \quad \vdash A \quad B \vdash D}{\vdash D} \wedge\text{-el}_{10}$$

$$\frac{\vdash A \quad \vdash B}{\vdash A \wedge B} \wedge\text{-in}_{11}$$

- Can be shown to be equivalent to the well-known constructive rules.
- These rules can be optimized to 3 rules.

Examples

Rules for \neg : 1 elimination rule and 1 introduction rule.

A	$\neg A$
0	1
1	0

Constructive:

$$\frac{\vdash \neg A \quad \vdash A}{\vdash D} \neg\text{-el} \qquad \frac{A \vdash \neg A}{\vdash \neg A} \neg\text{-in}^i$$

Classical:

$$\frac{\vdash \neg A \quad \vdash A}{\vdash D} \neg\text{-el} \qquad \frac{\neg A \vdash D \quad A \vdash D}{\vdash D} \neg\text{-in}^c$$



Lemma 1 to simplify the rules

$$\frac{\vdash A_1 \dots \vdash A_n \quad B_1 \vdash D \dots B_m \vdash D \quad C \vdash D}{\vdash D}$$

$$\frac{\vdash A_1 \dots \vdash A_n \quad \vdash C \quad B_1 \vdash D \dots B_m \vdash D}{\vdash D}$$

is equivalent to the system with these two rules replaced by

$$\frac{\vdash A_1 \dots \vdash A_n \quad B_1 \vdash D \dots B_m \vdash D}{\vdash D}$$

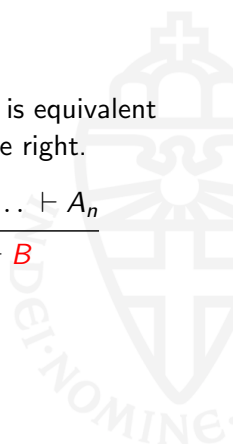


Lemma II to simplify the rules

A system with a deduction rule of the form to the left is equivalent to the system with this rule replaced by the rule on the right.

$$\frac{\vdash A_1 \dots \vdash A_n \quad B \vdash D}{\vdash D}$$

$$\frac{\vdash A_1 \dots \vdash A_n}{\vdash B}$$



The constructive connectives

We have already seen the \wedge, \neg rules. The optimized rules for \vee, \rightarrow, \top and \perp we obtain are:

$$\frac{\vdash A \vee B \quad A \vdash D \quad B \vdash D}{\vdash D} \vee\text{-el} \qquad \frac{\vdash A}{\vdash A \vee B} \vee\text{-in}_1 \qquad \frac{\vdash B}{\vdash A \vee B} \vee\text{-in}_2$$

$$\frac{\vdash A \rightarrow B \quad \vdash A}{\vdash B} \rightarrow\text{-el} \qquad \frac{\vdash B}{\vdash A \rightarrow B} \rightarrow\text{-in}_1 \qquad \frac{A \vdash A \rightarrow B}{\vdash A \rightarrow B} \rightarrow\text{-in}_2$$

$$\frac{}{\vdash \top} \top\text{-in} \qquad \frac{\vdash \perp}{\vdash D} \perp\text{-el}$$

The rules for the classical \rightarrow connective

$$\frac{\vdash A \rightarrow B \quad \vdash A}{\vdash B} \rightarrow\text{-el} \qquad \frac{\vdash B}{\vdash A \rightarrow B} \rightarrow\text{-in}_1 \qquad \frac{A \rightarrow B \vdash D \quad A \vdash D}{\vdash D} \rightarrow\text{-in}_2^c$$

Derivation of Peirce's law:

$$\frac{\frac{\frac{A \vdash A}{A \vdash ((A \rightarrow B) \rightarrow A) \rightarrow A} \quad \frac{\frac{\frac{(A \rightarrow B) \rightarrow A \vdash (A \rightarrow B) \rightarrow A \quad A \rightarrow B \vdash A \rightarrow B}{A \rightarrow B, (A \rightarrow B) \rightarrow A \vdash A}}{A \rightarrow B, (A \rightarrow B) \rightarrow A \vdash ((A \rightarrow B) \rightarrow A) \rightarrow A}}{A \rightarrow B \vdash ((A \rightarrow B) \rightarrow A) \rightarrow A} \rightarrow\text{-in}_2^c}{\vdash ((A \rightarrow B) \rightarrow A) \rightarrow A} \rightarrow\text{-in}_2^c$$

The “If Then Else” connective

Notation: $A \rightarrow B / C$ for if A then B else C .

p	q	r	$p \rightarrow q / r$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

The optimized constructive rules are:

$$\frac{\vdash A \rightarrow B / C \quad \vdash A}{\vdash B} \text{ then-el}$$

$$\frac{\vdash A \quad \vdash B}{\vdash A \rightarrow B / C} \text{ then-in}$$

$$\frac{\vdash A \rightarrow B / C \quad A \vdash D \quad C \vdash D}{\vdash D} \text{ else-el}$$

$$\frac{A \vdash A \rightarrow B / C \quad \vdash C}{\vdash A \rightarrow B / C} \text{ else-in}$$



Some facts about constructive “If Then Else”

$A \rightarrow B / C$ is logically equivalent to $(A \rightarrow B) \wedge (A \vee C)$

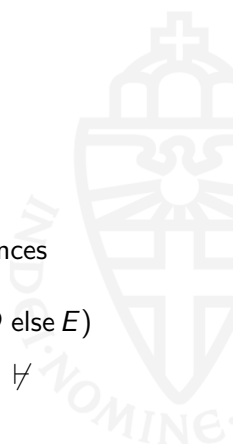
We have the well-known classical equivalence

$$\text{if } A \text{ then } B \text{ else } B \equiv B$$

We don't have the other well-known classical equivalences

if (if A then B else C) then D else E $\not\equiv$
if A then (if B then D else E) else (if C then D else E)

if A then (if B then D else E) else (if C then D else E) $\not\equiv$
if (if A then B else C) then D else E



“If Then Else” \rightarrow \top \perp is functionally complete

We can define the usual constructive connectives in terms of if-then-else, \top and \perp :

$$A \dot{\vee} B := A \rightarrow A/B \quad A \dot{\wedge} B := A \rightarrow B/A$$

$$A \dot{\rightarrow} B := A \rightarrow B/\top \quad \dot{\neg}A := A \rightarrow \perp/\top$$

LEMMA The defined connectives satisfy the original constructive deduction rules for these same connectives.

COROLLARY The constructive connective if-then-else, together with \top and \perp , is functionally complete.

Sheffer stroke or NAND connective [1]

The truth table for $\text{nand}(A, B)$, which we write as $A \uparrow B$ is as follows.

A	B	$A \uparrow B$
0	0	1
0	1	1
1	0	1
1	1	0

From this we derive the following optimized rules.

$$\frac{A \vdash A \uparrow B}{\vdash A \uparrow B} \uparrow\text{-inl}$$

$$\frac{B \vdash A \uparrow B}{\vdash A \uparrow B} \uparrow\text{-inr}$$

$$\frac{\vdash A \uparrow B \quad \vdash A \quad \vdash B}{\vdash D} \uparrow\text{-el}$$



Sheffer stroke or NAND connective [II]

The usual connectives can be defined in terms of nand.

$$\begin{aligned}\neg A &:= A \uparrow A \\ A \vee B &:= (A \uparrow A) \uparrow (B \uparrow B) \\ A \wedge B &:= (A \uparrow B) \uparrow (A \uparrow B) \\ A \rightarrow B &:= A \uparrow (B \uparrow B)\end{aligned}$$

This gives rise to an embedding $(-)^{\uparrow}$ of intuitionistic proposition logic \vdash_i into the nand-logic \vdash_{\uparrow} .

PROPOSITION For A a formula in proposition logic,

$$\vdash_i \neg\neg A \quad \Longleftrightarrow \quad \vdash_{\uparrow} (A)^{\uparrow}.$$



Kripke semantics for the constructive rules

For each n -ary connective c , we assume a truth table $t_c : \{0, 1\}^n \rightarrow \{0, 1\}$ and the defined constructive deduction rules.

DEFINITION A **Kripke model** is a triple (W, \leq, at) where W is a set of worlds, \leq a reflexive, transitive relation on W and a function $\text{at} : W \rightarrow \wp(\text{At})$ satisfying $w \leq w' \Rightarrow \text{at}(w) \subseteq \text{at}(w')$.

We define the notion **φ is true in world w** (usually written $w \Vdash \varphi$) by defining $\llbracket \varphi \rrbracket_w \in \{0, 1\}$

DEFINITION of $\llbracket \varphi \rrbracket_w \in \{0, 1\}$, by induction on φ :

- (atom) if φ is atomic, $\llbracket \varphi \rrbracket_w = 1$ iff $\varphi \in \text{at}(w)$.
- (connective) for $\varphi = c(\varphi_1, \dots, \varphi_n)$, $\llbracket \varphi \rrbracket_w = 1$ iff for each $w' \geq w$, $t_c(\llbracket \varphi_1 \rrbracket_{w'}, \dots, \llbracket \varphi_n \rrbracket_{w'}) = 1$ where t_c is the truth table of c .

$\Gamma \Vdash \psi :=$ for each Kripke model and each world w , if $\llbracket \varphi \rrbracket_w = 1$ for each φ in Γ , then $\llbracket \psi \rrbracket_w = 1$.

Kripke semantics for the constructive rules

LEMMA (Soundness) If $\Gamma \vdash \psi$, then $\Gamma \models \psi$

Proof. Induction on the derivation of $\Gamma \vdash \psi$.

For completeness we need to construct a special Kripke model.

- In the literature, the completeness of Kripke semantics is proved using *prime theories*.
- A theory is prime if it satisfies the **disjunction property**: if $\Gamma \vdash A \vee B$, then $\Gamma \vdash A$ or $\Gamma \vdash B$.
- We may not have \vee in our set of connective, and we may have others that “behave \vee -like”,
- (But we can generalize the disjunction property to arbitrary n -ary constructive connectives that are “splitting”.)
- We apply a kind of Lindenbaum construction (also used by Milne for the classical case).

Kripke semantics for the constructive rules

DEFINITION For ψ a formula and Γ a set of formulas, we say that Γ is ψ -maximal if

- $\Gamma \not\vdash \psi$ and
- for every formula $\varphi \notin \Gamma$ we have: $\Gamma, \varphi \vdash \psi$.

NB. Given ψ and Γ such that $\Gamma \not\vdash \psi$, we can extend Γ to a ψ -maximal set Γ' that contains Γ .

Simple important facts about ψ -maximal sets Γ :

- 1 For every φ , we have $\varphi \in \Gamma$ or $\Gamma, \varphi \vdash \psi$.
- 2 For every φ , if $\Gamma \vdash \varphi$, then $\varphi \in \Gamma$.



Completeness of Kripke semantics

DEFINITION We define the Kripke model $U = (W, \leq, \text{at})$:

- $W := \{(\Gamma, \psi) \mid \Gamma \text{ is a } \psi\text{-maximal set}\}$.
- $(\Gamma, \psi) \leq (\Gamma', \psi') := \Gamma \subseteq \Gamma'$.
- $\text{at}(\Gamma, \psi) := \Gamma \cap \text{At}$.

LEMMA In the model U we have, for all worlds $(\Gamma, \psi) \in W$:

$$\varphi \in \Gamma \iff \llbracket \varphi \rrbracket_{(\Gamma, \psi)} = 1 \quad (\forall \varphi)$$

Proof: Induction on the structure of φ .

THEOREM If $\Gamma \models \psi$, then $\Gamma \vdash \psi$.

Proof. Suppose $\Gamma \models \psi$ and $\Gamma \not\vdash \psi$. Then we can find a ψ -maximal superset Γ' of Γ such that $\Gamma' \not\vdash \psi$. In particular: ψ is not in Γ' . So (Γ', ψ) is a world in the Kripke model U in which each member of Γ is true, but ψ is not. Contradiction to $\Gamma \models \psi$, so $\Gamma \vdash \psi$.

Some general proof-theoretic properties

The n -ary connective c is **i, j -splitting** in case

$$t_c(p_1, \dots, p_{i-1}, 0, p_{i+1}, \dots, p_{j-1}, 0, p_{j+1}, \dots, p_n) = 0$$

for all $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_{j-1}, p_{j+1}, \dots, p_n \in \{0, 1\}$.

LEMMA For c an i, j -splitting connective, if $\vdash c(A_1, \dots, A_n)$, then $\vdash A_i$ or $\vdash A_j$.

For example: if $\vdash A \rightarrow B / C$, then $\vdash A$ or $\vdash C$. (And also: if $\vdash A \rightarrow B / C$, then $\vdash B$ or $\vdash C$.)

An n -ary connective c is **monotonic** if $t_c : \{0, 1\}^n \rightarrow \{0, 1\}$ is monotonic under the ordering induced by $0 \leq 1$.

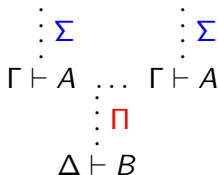
LEMMA For c monotonic, the classical and constructive derivation rules are equivalent.

LEMMA For c_1, c_2 non-monotonic, if we take the classical rules for c_1 and the constructive rules for c_2 , we can derive the classical rules for c_2 .

Substituting a deduction in another

LEMMA: If $\Gamma \vdash A$ and $\Delta, A \vdash B$, then $\Gamma, \Delta \vdash B$

If Σ is a deduction of $\Gamma \vdash A$ and Π is a deduction of $\Delta, A \vdash B$, then we have the following deduction of $\Gamma, \Delta \vdash B$:



In Π , every application of an (axiom) rule at a leaf, deriving $\Delta' \vdash A$ for some $\Delta' \supseteq \Delta$ is replaced by a copy of a deduction Σ , which is also a deduction of $\Delta', \Gamma \vdash A$.

Detours (cuts) in constructive logic

Remember that the rules for c arise from rows in the truth table t_c :

A_1	\dots	A_n	$c(A_1, \dots, A_n)$
p_1	\dots	p_n	0
q_1	\dots	q_n	1

DEFINITION A **detour convertibility** is a pattern of the following form, where $\Phi = c(A_1, \dots, A_n)$.

$$\frac{\dots \frac{\boxed{\Sigma_j}}{\Gamma \vdash A_j} \dots \frac{\boxed{\Sigma_i}}{\Gamma, A_i \vdash \Phi} \dots}{\Gamma \vdash \Phi} \text{ in} \quad \frac{\dots \frac{\boxed{\Pi_k}}{\Gamma \vdash A_k} \dots \frac{\boxed{\Pi_\ell}}{\Gamma, A_\ell \vdash D} \dots}{\Gamma \vdash D} \text{ el}$$

- $q_j = 1$ for A_j and $q_i = 0$ for A_i
- $p_k = 1$ for A_k and $p_\ell = 0$ for A_ℓ

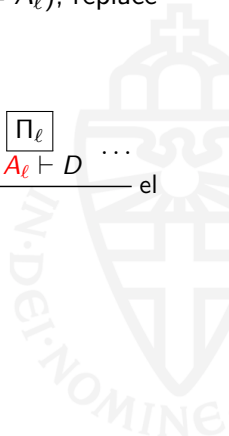
Eliminating a detour (detour conversion) (I)

The *elimination of a detour* is defined by replacing the deduction pattern by another one. If $j = \ell$ (for some j, ℓ , so $A_j = A_\ell$), replace

$$\frac{\begin{array}{c} \dots \quad \boxed{\Sigma_j} \quad \dots \quad \boxed{\Sigma_i} \quad \dots \\ \Gamma \vdash A_j \quad \dots \quad \Gamma, A_i \vdash \Phi \quad \dots \end{array}}{\Gamma \vdash \Phi} \text{ in} \quad \frac{\begin{array}{c} \dots \quad \boxed{\Pi_k} \quad \dots \quad \boxed{\Pi_\ell} \quad \dots \\ \Gamma \vdash A_k \quad \dots \quad \Gamma, A_\ell \vdash D \quad \dots \end{array}}{\Gamma \vdash D} \text{ el}$$

by

$$\frac{\begin{array}{c} \boxed{\Sigma_j} \quad \dots \quad \boxed{\Sigma_j} \\ \Gamma \vdash A_j \quad \dots \quad \Gamma \vdash A_j \end{array}}{\boxed{\Pi_\ell} \\ \Gamma \vdash D}$$



Eliminating a detour (detour conversion) (II)

If $i = k$ (for some i, k , so $A_i = A_k$), replace

$$\frac{\dots \frac{\boxed{\Sigma_j}}{\Gamma \vdash A_j} \dots \frac{\boxed{\Sigma_i}}{\Gamma, A_i \vdash \Phi} \dots}{\Gamma \vdash \Phi} \text{ in} \quad \dots \frac{\boxed{\Pi_k}}{\Gamma \vdash A_k} \dots \frac{\boxed{\Pi_\ell}}{\Gamma, A_\ell \vdash D} \dots}{\Gamma \vdash D} \text{ el}$$

by

$$\frac{\frac{\frac{\boxed{\Pi_k}}{\Gamma \vdash A_k} \dots \frac{\boxed{\Pi_k}}{\Gamma \vdash A_k}}{\boxed{\Sigma_i}}{\Gamma \vdash \Phi} \quad \dots \frac{\boxed{\Pi_k}}{\Gamma \vdash A_k} \dots \frac{\boxed{\Pi_\ell}}{\Gamma, A_\ell \vdash D} \dots}{\Gamma \vdash D} \text{ el}$$

Observation

$$\frac{\dots \quad \boxed{\Sigma_j} \quad \Gamma \vdash A_j \quad \dots \quad \boxed{\Sigma_i} \quad \Gamma, A_i \vdash \phi \quad \dots}{\Gamma \vdash \phi} \text{ in} \quad \dots \quad \boxed{\Pi_k} \quad \Gamma \vdash A_k \quad \dots \quad \boxed{\Pi_\ell} \quad \Gamma, A_\ell \vdash D \quad \dots}{\Gamma \vdash D} \text{ el}$$

- There can be several “matching” (i, k) or (j, ℓ) pairs.
- So: detour conversion (“ β -rule”) is non-deterministic in general.

Permutation conversion

The **permutation conversion** is defined by replacing the derivation pattern on the previous slide by

$$\frac{\vdash \Psi \dots \vdash A_j \quad \dots \quad \frac{\vdots \boxed{\Sigma_j} \quad \vdots \boxed{\Sigma_i} \quad A_i \vdash \Phi \quad \dots \quad \vdots \boxed{\Pi_k} \quad A_i \vdash B_k \quad \dots \quad \vdots \boxed{\Pi_\ell} \quad A_i, B_\ell \vdash D \quad \dots}{A_i \vdash D} \text{el}_{r'} \quad \text{el}_r}{\vdash D}$$

This gives rise to copying of sub-derivations: for every A_i we copy the sub-derivations Π_1, \dots, Π_n .

Curry-Howard proofs-as-terms

We define rules for the judgment $\Gamma \vdash t : A$, where

- A is a formula,
- Γ is a set of declarations $\{x_1 : A_1, \dots, x_m : A_m\}$, where the A_i are formulas and the x_i are term-variables,
- t is a **proof-term**:

$$t ::= x \mid \{\bar{t}; \overline{\lambda x : A.t}\}_r \mid t \cdot_r [\bar{t}; \overline{\lambda x : A.t}]$$

where x ranges over variables and r ranges over the rules.

For a connective $c \in \mathcal{C}$, r an introduction rule for c and r' an elimination rule for c , we have

- an **introduction term** $\{\bar{t}; \overline{\lambda x : A.t}\}_r$
- an **elimination term** $t \cdot_{r'} [\bar{t}; \overline{\lambda x : A.t}]$

Curry-Howard typing rules

Let $\Phi = c(A_1, \dots, A_n)$ and r a rule for c .

$$\begin{array}{c}
 \frac{}{\Gamma \vdash x_i : A_i} \text{ if } x_i : A_i \in \Gamma \\
 \\
 \frac{\dots \Gamma \vdash p_j : A_j \dots \quad \dots \Gamma, y_i : A_i \vdash q_i : \Phi \dots}{\Gamma \vdash \{\bar{p} ; \overline{\lambda y : A.q}\}_r : \Phi} \text{ in} \\
 \\
 \frac{\Gamma \vdash t : \Phi \quad \dots \Gamma \vdash p_k : A_k \dots \quad \dots \Gamma, y_\ell : A_\ell \vdash q_\ell : D}{\Gamma \vdash t \cdot_r [\bar{p} ; \overline{\lambda y : A.q}] : D} \text{ el}
 \end{array}$$

Here, \bar{p} is the sequence of terms $p_1, \dots, p_{m'}$ for all the 1-entries in rule r of the truth table, and $\overline{\lambda y : A.q}$ is the sequence of terms $\lambda y_1 : A_1.q_1, \dots, \lambda y_m : A_m.q_m$ for all the 0-entries in r .

Reductions on terms for detours

Term reduction rules that correspond to **detour conversions**.

- For simplicity we write the “matching cases” as last term of the sequence.
- For the $j = \ell$ case, that is, $p_j : A_j$ and $y_\ell : A_\ell$ with $A_j = A_\ell$:

$$\{\overline{p}, \overline{p_j} ; \overline{\lambda x. q}\} \cdot [\overline{s} ; \overline{\lambda y. r}, \overline{\lambda y_\ell. r_\ell}] \longrightarrow_a r_\ell[y_\ell := p_j]$$

- For the $i = k$ case, that is, $x_i : A_i$ and $s_k : A_k$ with $A_i = A_k$:

$$\{\overline{p} ; \overline{\lambda x. q}, \overline{\lambda x_i. q_i}\} \cdot [\overline{s}, \overline{s_k} ; \overline{\lambda y. r}] \longrightarrow_a q_i[x_i := s_k] \cdot [\overline{s}, \overline{s_k} ; \overline{\lambda y. r}]$$

$\overline{p}, \overline{p_j}$ should be understood as a sequence $p_1, \dots, p_j, \dots, p_{m'}$, where the p_j that matches the r_ℓ in $\overline{\lambda y. r}, \overline{\lambda y_\ell. r_\ell}$ has been singled out.

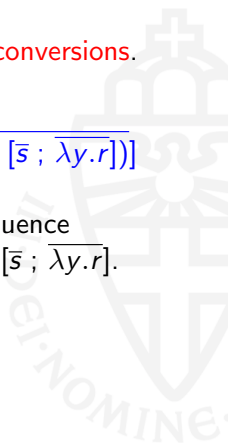
NB There is always (at least one) **matching case**, because intro/elim rules comes from different lines in the truth table.

Reductions on terms for permutations

We add the following reduction rules for **permutation conversions**.

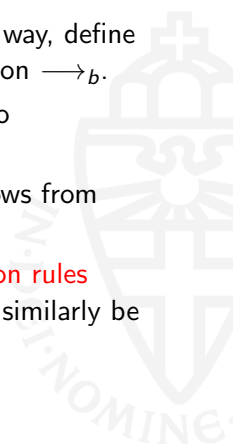
$$(t \cdot_r [\overline{p}; \overline{\lambda x. q}]) \cdot_{r'} [\overline{s}; \overline{\lambda y. r}] \longrightarrow_b t \cdot_r [\overline{p}; \overline{\lambda x. (q \cdot_{r'} [\overline{s}; \overline{\lambda y. r}])}]$$

Here, $\overline{\lambda x. (q \cdot_{r'} [\overline{s}; \overline{\lambda y. r}])}$ should be understood as a sequence $\lambda x_1. q_1, \dots, \lambda x_m. q_m$ where each q_j is replaced by $q_j \cdot_{r'} [\overline{s}; \overline{\lambda y. r}]$.



Optimized reductions on optimized terms

- On optimized terms, one can also, in a canonical way, define detour conversion \longrightarrow_a and permutation conversion \longrightarrow_b .
- Detour reduction on optimized terms translates to (multi-step) detour reduction on the full terms.
- So, strong normalization on optimized terms follows from strong normalization on full terms.
- Other well-known rules, like the **general elimination rules** studied by Schroeder-Heister and Von Plato, can similarly be translated to our full rules.



Normalization

THEOREM The reduction \longrightarrow_b is strongly normalizing

$$(t \cdot_r [\bar{p}; \overline{\lambda x. q}]) \cdot_{r'} [\bar{s}; \overline{\lambda y. r}] \longrightarrow_b t \cdot_r [\bar{p}; \overline{\lambda x. (q \cdot_{r'} [\bar{s}; \overline{\lambda y. r}])}]$$

PROOF The measure $| - |$ decreases with every reduction step.

$$\begin{aligned} |x| &:= 1 \\ |\{\bar{p}; \overline{\lambda y. q}\}| &:= \sum |p_i| + \sum |q_j| \\ |t \cdot [\bar{s}; \overline{\lambda y. u}]| &:= |t|(2 + \sum |s_k| + \sum |u_\ell|) \end{aligned}$$



Normalization

THEOREM The reduction \longrightarrow_a is strongly normalizing.

$$\{\overline{p}, \overline{p_j}; \overline{\lambda x. q}\} \cdot [\overline{s}; \overline{\lambda y. r}, \overline{\lambda y_\ell. r_\ell}] \longrightarrow_a r_\ell[y_\ell := p_j]$$

(for the $A_j = A_\ell$ case, $p_j : A_j$ and $y_\ell : A_\ell$ with $A_j = A_\ell$)

$$\{\overline{p}; \overline{\lambda x. q}, \overline{\lambda x_i. q_i}\} \cdot [\overline{s}, \overline{s_k}; \overline{\lambda y. r}] \longrightarrow_a q_i[x_i := s_k] \cdot [\overline{s}, \overline{s_k}; \overline{\lambda y. r}]$$

(for the $A_i = A_k$ case, $x_i : A_i$ and $s_k : A_k$ with $A_i = A_k$)

PROOF We adapt the saturated sets method of Tait.

COROLLARY the combination \longrightarrow_{ab} is weakly normalizing.
Basically: take the \longrightarrow_b -normal-form and then contract the innermost \longrightarrow_a -redex of **highest rank**. (This generalizes the Gandy-Turing WN proof for simple type theory, $\lambda \rightarrow$.)

Strong Normalization

We have obtained a proof of Strong Normalization for general IPC_C .

Rough outline of the proof (generalizing a proof of SN for IPC by Philippe De Groot):

- Define a “double negation” translation from IPC_C formulas to $\lambda \rightarrow$ -types.
- Define a reduction preserving “CPS” translation from IPC_C terms to $\lambda \rightarrow$ -parallel.
($\lambda \rightarrow$ extended with $[M_1, \dots, M_n] : A$ if $M_i : A$ for $1 \leq i \leq n$.)
- Prove SN for $\lambda \rightarrow$ -parallel.

$\lambda \rightarrow$ -parallel

- Types: $\sigma ::= o \mid (\sigma \rightarrow \sigma)$
- Terms: $M ::= x \mid (M M) \mid (\lambda x.M) \mid [M_1, \dots, M_n]$ ($n > 1$).
- Typing rules

$$\frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \rightarrow B}$$

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A} \qquad \frac{\Gamma \vdash M_1 : A \quad \dots \quad \Gamma \vdash M_n : A}{\Gamma \vdash [M_1, \dots, M_n] : A}$$

- Reduction rules: $(\lambda x.M) N \rightarrow_{\beta} M[x := N]$ plus

$$[M_1, \dots, M_n] N \rightarrow_{\beta} [M_1 N, \dots, M_n N]$$

SN can be proved by adapting the well-known Tait proof.

Translating formulas to types (outline)

Abbreviate $\neg A := A \rightarrow o$.

- For a proposition letter, $\widehat{A} := \neg\neg A$.
- For $\Phi = c(A_1, \dots, A_n)$ with elimination rules r_1, \dots, r_t

$$\widehat{\Phi} := \neg(E_1 \rightarrow \dots \rightarrow E_t \rightarrow o),$$

where

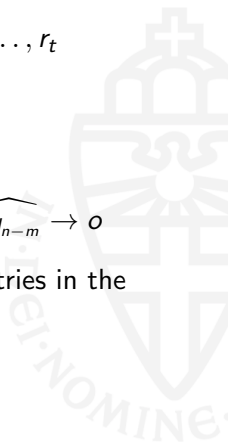
$$E_s := \widehat{A}_{k_1} \rightarrow \dots \rightarrow \widehat{A}_{k_m} \rightarrow \neg\widehat{A}_{l_1} \rightarrow \dots \rightarrow \neg\widehat{A}_{l_{n-m}} \rightarrow o$$

with the A_k the 1-entries and the A_l are the 0-entries in the truth table.

For example

$$\widehat{A \wedge B} = \neg(\neg\neg\widehat{A} \rightarrow \neg\neg\widehat{B} \rightarrow o)$$

$$\widehat{A \vee B} = \neg((\neg\widehat{A} \rightarrow \neg\widehat{B} \rightarrow o) \rightarrow o)$$



Translating proof-terms to $\lambda \rightarrow$ -parallel terms (outline)

We have a translation \widehat{M} and a second translation $\widehat{\widehat{M}}$. (This is a generalization of the CPS translation $\overline{\overline{M}}$ of Plotkin, that De Groot also uses.)

We can prove

- If $M \rightarrow_b N$, then $\widehat{M} = \widehat{N}$
- If $\widehat{\widehat{M}} \subset K$ ($\widehat{\widehat{M}}$ is a **subterm** of K), then

$$\begin{array}{ccccc}
 M & \mapsto & \widehat{M} & \subset & K \\
 \downarrow a & & & & \downarrow \beta + \\
 N & \mapsto & \widehat{\widehat{N}} & \subset & \exists K'
 \end{array}$$

From this we derive Strong Normalization.



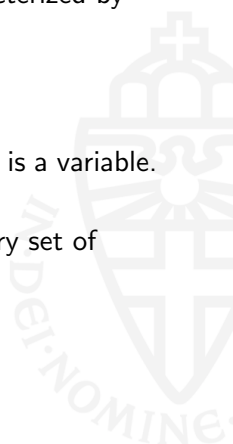
Consequences of Normalization

The set of **terms in normal form** of $\text{IPC}_{\mathcal{C}}$, NF is characterized by the following inductive definition.

- $x \in \text{NF}$ for every variable x ,
- $\{\bar{p} ; \overline{\lambda y. q}\} \in \text{NF}$ if all p_i and q_j are in NF,
- $x \cdot [\bar{p} ; \overline{\lambda y. q}] \in \text{NF}$ if all p_i and q_j are in NF and x is a variable.

As corollaries of Normalization we have, for an arbitrary set of connectives:

- subformula property
- consistency of the logic
- decidability of the logic



Classical logic

For classical logic, we have:

$$\frac{A_1 \quad \dots \quad A_n \mid \Phi}{p_1 \quad \dots \quad p_n \mid 0} \mapsto \frac{\vdash \Phi \dots \vdash A_j \text{ (if } p_j = 1) \dots A_i \vdash D \text{ (if } p_i = 0) \dots}{\vdash D} \text{el}$$

classical intro

$$\frac{A_1 \quad \dots \quad A_n \mid \Phi}{r_1 \quad \dots \quad r_n \mid 1} \mapsto \frac{\Phi \vdash D \dots \vdash A_j \text{ (if } r_j = 1) \dots A_i \vdash D \text{ (if } r_i = 0) \dots}{\vdash D} \text{in}$$

- If $p_j = 1$ (or $r_j = 1$) in t_c , then A_j occurs as Lemma in the rule
- If $p_j = 0$ (or $r_j = 0$) in t_c , then A_i occurs as Casus in the rule

We call $\vdash \Phi$ (resp. $\Phi \vdash D$) the **major premise** and the other hypotheses of the rule we call the **minor premises**.

Proof terms for classical logic

$$t ::= x \mid (\lambda y : A.t) \star_r \{\bar{t}; \overline{\lambda x : A.t}\} \mid t \cdot_r [\bar{t}; \overline{\lambda x : A.t}]$$

where x ranges over variables and r ranges over the rules of all the connectives.

The terms are typed using the following derivation rules.

$\frac{}{\Gamma \vdash x_i : A_i} \text{ if } x_i : A_i \in \Gamma$
$\frac{\Gamma, z : \Phi \vdash t : D \dots \Gamma \vdash p_i : A_i \dots \quad \dots \Gamma, y_j : A_j \vdash q_j : D \dots}{\Gamma \vdash (\lambda z : \Phi.t) \star_r \{\bar{p}; \overline{\lambda y : A.q}\} : D} \text{ in}$
$\frac{\Gamma \vdash t : \Phi \quad \dots \Gamma \vdash p_k : A_k \dots \quad \dots \Gamma, y_\ell : A_\ell \vdash q_\ell : D}{\Gamma \vdash t \cdot_r [\bar{p}; \overline{\lambda y : A.q}] : D} \text{ el}$

Reduction for proof terms in classical logic

- First perform **permutation reductions**.
- Then we perform **detour reductions**.

This is similar to the constructive case, except for now

- a term is in **permutation normal form** if all lemmas are variables,
- a **detour** is an elimination of Φ followed by an introduction of ϕ .

NB: in constructive logic, a “detour” is an introduction **directly followed** by an elimination. Here it is the other way around, and the introduction need not follow the elimination directly.

This is the abstract syntax N for **permutation normal forms**:

$$N ::= x \mid (\lambda y : A.N) \star \{\bar{z} ; \overline{\lambda x : A.N}\} \mid y \cdot [\bar{z} ; \overline{\lambda x : A.N}],$$

where x, y, z range over variables.

Detours for proof terms in classical logic

A **detour** is a pattern of the following shape

$$(\lambda x : \Phi \dots (x \cdot [\bar{v} ; \overline{\lambda w : A.s}]) \dots) \star \{\bar{z} ; \overline{\lambda y : A.q}\}$$

that is, an elimination of $\Phi = c(A_1, \dots, A_n)$ followed by an introduction of Φ , with an arbitrary number of steps in between.

For terms in permutation normal form, detours can be eliminated, obtaining a term in normal form which satisfies the **sub-formula property**.

Notes to the pattern of a detour:

- the indicated occurrence need not be the only occurrence of x
- variable x may not occur at all; that is the simplest situation.

Eliminating detours

Eliminating detours is done by the following reduction steps:

- $$(\lambda x : \Phi \dots (x \cdot [\bar{v} ; \overline{\lambda w : A.s}]) \dots) \star \{\bar{z} ; \overline{\lambda y : A.q}\} \longrightarrow_a$$

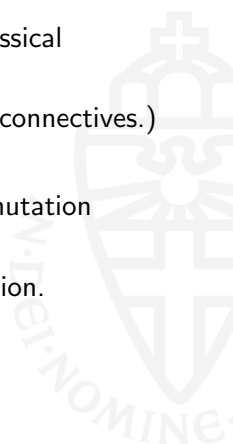
$$(\lambda x : \Phi \dots (s_\ell[w_\ell := z_i]) \dots) \star \{\bar{z} ; \overline{\lambda y : A.q}\}$$
 if $i = \ell (A_i = A_\ell)$ is a “matching case” for the subformulas of Φ .
- $$(\lambda x : \Phi \dots (x \cdot [\bar{v} ; \overline{\lambda w : A.s}]) \dots) \star \{\bar{z} ; \overline{\lambda y : A.q}\} \longrightarrow_a$$

$$(\lambda x : \Phi \dots (q_j[y_j := v_k]) \dots) \star \{\bar{z} ; \overline{\lambda y : A.q}\}$$
 if $j = k (A_j = A_k)$ is a “matching case” for the subformulas of Φ .
- $$(\lambda x : \Phi.t) \star \{\bar{z} ; \overline{\lambda y : A.q}\} \longrightarrow_a t \quad \text{if } x \notin \text{FV}(t).$$

Tonny Hurkens has given a proof that this normalizes

Conclusions

- Simple general way to derive constructive and classical deduction rules for (new) connectives.
- Study connectives “in isolation”. (Without other connectives.)
- Generic Kripke semantics for constructive logic
- General definitions of detour conversion and permutation conversion.
- General Curry-Howard proofs-as-terms interpretation.
- General Strong Normalization proof.



Future work and Related

- Meaning of the new connectives as inductive data types.
- Study conditions for the set of rules to be Church-Rosser.
- Study the computational meaning of classical proof terms.
- Relation with other well-known term calculi for classical logic: subtraction logic (Crolard), $\lambda\mu$ (Parigot), $\bar{\lambda}\mu\tilde{\mu}$ (Curien, Herbelin).

Related work:

- Dyckhoff; Milne; von Plato and Negri; Schroeder-Heister; Joachimski and Matthes; Baaz, Fermüller and Zach; Abel; ...
- “Harmony” in logic (following Prawitz)

Questions?

