On Relating Indexed W-Types with Ordinary Ones

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Introduction Indexed W-types model inductive families, just as ordinary W-types model inductive types. In the context of extensional type theory, indexed W-types were shown constructible from ordinary ones by Gambino and Hyland in a past TYPES post-proceedings [3] using equivalent categorical terms: they construct initial algebras for dependent polynomial functors from non-dependent ones in a locally cartesian closed category. In intensional type theory with function extensionality[1], an analogous result should hold when considering the corresponding homotopified notion [1] of (indexed) W-types.[2]

Though tedious, this is provable using ad-hoc term-level manipulations following essentially the idea from the extensional case. Instead, we want to highlight a conceptually clean alternative. By illuminating a deeper categorical nature of the extensional construction [3], we make it amendable to higher categorical generalization in terms of locally cartesian closed quasi-categories.

Recent work, partially in progress, by Szumiło [5] and Kapulkin exhibits the syntax of intensional type theory with function extensionality as a locally cartesian closed quasi-category. After verifying that quasi-categorical notions like initial objects in algebra quasi-categories agree with their counterparts defined in the internal language of type theory, this should prove the desired result.

It is noteworthy this approach lets us leave the realm of type-theoretic syntax by working in the semantic domain of quasi-categories. It does not seem possible to formalize internally the infinite tower of coherence e.g. of the notion of algebra morphisms with their compositional structure. Nor is it needed: as realized early on in homotopy type theory, contractibility is internally expressible, letting us define notions like homotopy initial algebra by referencing only the first few levels [1]. Hence, only finitely many levels of coherence will be needed at any point.

However, several steps in a type-theoretic proof would each require explicating an additional layer of coherence to start with, making a manual translation to an internal proof rather infeasable and unreadable (and of little conceptual value).[3] This is due to a deficiency of current syntax for homotopy type theory to adequately capture higher-dimensional categorical coherence in a way that is comparable to how the identity type captures higher-dimensional groupoidal coherence.

Work in progress.

Sketch: the extensional case Since our main contribution is introducing abstraction to a previously concrete argument [3], the use of categorical language in the following sketch is unavoidable. However, we only want to give some intuition for the underlying ideas. We have retricted us to the extensional setting, and the reader is invited to take from it whatever they want.

[1] That is: homotopy type theory without requiring a universe.

[2] Computation and uniqueness laws are formulated using coherent propositional equality, making them homotopy initial algebras.

[3] We conjecture that an effective proof term is generatable from the quasi-categorical proof.

Let $\mathcal{C}$ be a locally cartesian closed. Given $f : B \to A$ and $s : B \to I$ and $t : A \to I$, the associated dependent polynomial endofunctor $[\![f_{s,t}]\!] : \mathcal{C}/I \to \mathcal{C}/I$ is composed of base change along s, dependent product along f, and dependent sum along t:

$$\mathcal{C}/I \xrightarrow{\Delta_s} \mathcal{C}/B \xrightarrow{\Pi_f} \mathcal{C}/A \xrightarrow{\Sigma_t} \mathcal{C}/I$$

The non-dependent version arises as the special case $I = 1$, in which case we will omit s and t.

The basic idea of [3] for constructing the initial algebra $\mu F = \mu\,[\![f_{s,t}]\!]$ is to carve it out of $\mu\,[\![f]\!]$, which we regard as the type of well-founded trees possibly ill-typed with respect to the I-indexing. This is done by taking the equalizer of the diagram $\mu\,[\![f]\!] \rightrightarrows \mu\,[\![I \times f]\!]$ where the two maps copy to each node the index expected from "below" and given by "above", respectively; taking the equalizer corresponds to type-checking the indexing information. The construction of these maps is very much hands-on via explicit recursive definitions, with a lack of symmetry between the two maps and an ad-hoc choice for the index value at the root of the "below" map. This suggests that conceptually we ought to be working in a different slice.

Abbreviate $F = [\![f_{s,t}]\!]$ and write $[\![f]\!] \cong \Sigma_I F \Delta_I$ and $[\![I \times f]\!] \cong \Sigma_I \Delta_I \Sigma_I F \Delta_I$ using 2-functoriality and Beck-Chevalley conditions. Abbreviating $T = \Delta_I \Sigma_I$, the rolling rule [2] allows to derive initial algebras $\mu(TF)$ from $\mu\,[\![f]\!]$ and $\mu(T^2F)$ from $\mu\,[\![I \times f]\!]$. Note T is a cartesian monad (T, η, θ). Having "rolled around" Δ_I, the two maps above are now given functorially: we define the candidate carrier X for μF simply by the following (coreflexive) equalizer diagram:[4]

$$X \dashrightarrow \mu(TF) \underset{\mu(\eta TF)}{\overset{\mu(T\eta F)}{\rightrightarrows}} \mu(T^2F), \qquad \mu(TF) \xleftarrow{\mu(\theta F)} \mu(T^2F)$$

Observe that this is in complete analogy to the (coreflexive) equalizer diagram

$$F \xrightarrow{\eta F} TF \underset{\eta TF}{\overset{T\eta F}{\rightrightarrows}} T^2F, \qquad TF \xleftarrow{\theta F} T^2F$$

induced by T since it is cartesian. The algebra structure $h : F(X) \to X$ is induced by coreflexive equalizers being cosifted limits together with preservation properties making the bifunctor $(H, Y) \mapsto H(Y)$ preserve coreflexive equalizers jointly. Initiality of (X, h) can then be seen to transfer from initiality of $\mu(TF)$ using an abstract fibrational argument.

References

[1] S. Awodey, N. Gambino, and K. Sojakova. Inductive types in homotopy type theory. In *Proc. of LICS '12*, pp. 95–104. IEEE CS, 2012.

[2] R. Backhouse, M. Bijsterveld, R. van Geldrop, and J. Van Der Woude. Category theory as coherently constructive lattice theory. Working document, 1998.

[3] N. Gambino and M. Hyland. Wellfounded trees and dependent polynomial functors. In *Revised Selected Papers from TYPES 2003*, v. 3085 of *Lect. Notes in Comput. Sci.*, pp. 210–225. Springer, 2004.

[4] N. Ghani, P. Hancock, C. McBride, and P. Morris. Indexed containers. Journal version draft, 2013.

[5] K. Szumiło. Two models for the homotopy theory of cocomplete homotopy theories. arXiv:1411.0303, 2014.

[4] It is an artifact of the 1-categorical setting that coreflexive equalizers are special cases of equalizers. In the higher dimensional context, they represent entirely different concepts. But even in the 1-categorical setting, coreflexive equalizers are set apart by being cosifted limits as used above. This confusion might have led to the unnecessary assumption of uniqueness of identity proofs (UIP) in certain formalizations [4].