The Next 700 Modal Type Assignment Systems

Andreas Abel
Talk presented by Andrea Vezzosi

Department of Computer Science and Engineering
Chalmers and Gothenburg University, Sweden

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Motivation

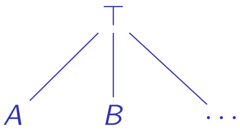
- Grand goal: integrate analyses (positivity, termination, strictness) into dependent type theory.
- Reason: with meta variables, type checking not cleanly separable from other analyses.
- A lot of type-based analyses out there:
  - Strictness.
  - Relevance.
  - Linearity.
  - Positivity.
- Can be formulated as non-standard type systems.
- Sometimes a bit ad-hoc.
- Can we systematize them?

Simply Typed Lambda-Calculus With Largest Type

- Terms $t, u ::= x \mid \lambda x t \mid t\, u$ (untyped).
- Type assignment $\Gamma \vdash t : A$.
- Types $A, B ::= \top \mid A \to B$.
- Contexts Γ are *total* functions from term variables to types.
  - Empty context ε is constant function $\varepsilon(x) = \top$.
  - Domain $\mathrm{dom}(\Gamma) = \{x \mid \Gamma(x) \neq \top\}$.
  - Update/extension $\Gamma' = (\Gamma, x{:}A)$ is $\Gamma'(y) = \begin{cases} U & \text{if } y = x \\ \Gamma(y) & \text{otherwise} \end{cases}$
  - Singleton context $x{:}A$ is $(\varepsilon, x{:}A)$.

Trivial subtyping

- Subtyping $A \leq B$ iff $A = B$ or $B = \top$.

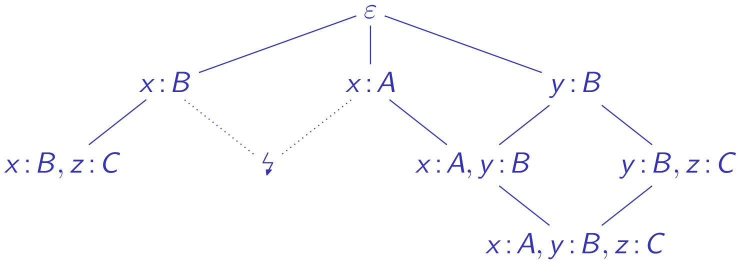


- Greatest lower bound $A \wedge B$ is undefined except that

$$A \wedge \top = \top \wedge A = A \wedge A = A.$$

Context subsumption = Weakening

- Subsumption $\Gamma \leq \Delta$ is pointwise: $\forall x.\ \Gamma(x) \leq \Delta(x)$.



- Joining contexts $\Gamma \wedge \Delta$ is pointwise (undef. if undef. at some x):

$$(\Gamma \wedge \Delta)(x) = \Gamma(x) \wedge \Delta(x)$$

Note: $\varepsilon \wedge \Gamma = \Gamma \wedge \varepsilon = \Gamma \wedge \Gamma = \Gamma$.

Simple Type Assignment

$$\frac{}{x : A \vdash x : A}\ hyp \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x t : A \to B}\ abs$$

$$\frac{\Gamma \vdash t : A \to B \qquad \Delta \vdash u : A}{\Gamma \wedge \Delta \vdash t\, u : B}\ app$$

$$\frac{\Gamma \leq \Delta \qquad \Delta \vdash t : A \qquad A \leq B}{\Gamma \vdash t : B}\ sub$$

- Looks *linear*.
- But *app* allows contraction.
- And *sub* allows weakening.

Exact Quantitative Typing

- Quantifies resource use exactly.
- Quantified type $Q ::= qA$, quantity $p, q \in \mathbb{N}$.
- Types $A, B ::= \top \mid Q \to B$.
- E.g. $5A \to B$: *to produce one* B, *we need exactly 5* A.
- Context Γ maps variables x to quantified types Q.
- Scaling $p(qA) = (pq)A$ and $(p\Gamma)(x) = p(\Gamma(x))$.
- Partial sum $pA + qB = (p + q)(A \wedge B)$.
- Pointwise context sum $(\Gamma + \Delta)(x) = \Gamma(x) + \Delta(x)$.

Exact Quantitative Type Assignment

$$\frac{}{x : 1A \vdash x : A}\ hyp \qquad \frac{\Gamma, x : qA \vdash t : B}{\Gamma \vdash \lambda x t : qA \to B}\ abs$$

$$\frac{\Gamma \vdash t : qA \to B \qquad \Delta \vdash u : qA}{\Gamma + \Delta \vdash t\, u : B}\ app$$

$$\frac{\Gamma \vdash t : A}{q\Gamma \vdash t : qA}\ mod$$

- *mod* introduces quantified types.
- *app* splits resources between function and argument.

Wasting (Weakening)

- Subtyping quantified types.

$$\frac{q \geq q' \qquad A \leq A'}{qA \leq q'A'}$$

- Weakening.

$$\frac{\Gamma \leq \Delta \qquad \Delta \vdash t : A \qquad A \leq B}{\Gamma \vdash t : B}\ sub$$

Typical Quantitative Type Systems

- Strictness: Is a variable used *at least once*?
  Quantities: $0..$ and $1..$
- Relevance: Is a variable used *never*?
  Quantities: 0 and $1..$
- Linear typing: Is a variable used *exactly once* or *unrestrictedly*?
  Quantities: 1 and $0..$

Quantitative Typing, Revisited

- Quantities $q \in P$ are *sets* of natural numbers.
- $P \subseteq \mathcal{P}(\mathbb{N})$ is partially ordered by

$$p \leq q \iff p \supseteq q$$

- P should form a monoid with composition $pq = \{mn \mid m \in p \text{ and } n \in q\}$ and a suitable unit $\mathbb{1}$.

$$\frac{}{x : \mathbb{1}A \vdash x : A}\ hyp$$

- Sum $p + q$ is smallest set $r \in P$ such that $m + n \in r$ for all $m \in p$ and $n \in q$ (might not exists).
- Default element p_0 for empty context $\varepsilon(x) = p_0 \top$ (controls weakening).

Example 1: Linear typing

- $P = \{\mathbb{1}, !\}$ with $\mathbb{1} = \{1\}$ (linear) and $! = \mathbb{N}$ (unrestricted).

pq	$\mathbb{1}$	$\mathbb{N}$
$\mathbb{1}$	$\mathbb{1}$	$\mathbb{N}$
$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$

$p+q$	$\mathbb{1}$	$\mathbb{N}$
$\mathbb{1}$	/	$\mathbb{N}$
$\mathbb{N}$	$\mathbb{N}$	$\mathbb{N}$

- $\mathbb{1} + \mathbb{1}$ undefined: no contraction for linear hypotheses!
- $p_0 = \, ! \leq \mathbb{1}$: weakening only with unrestricted hypotheses.

$$(x : !A) \leq (x : !\top) = \varepsilon$$

- Hypothesis rule usable for $!$ via *sub*:

$$\frac{(x{:}!A) \leq (x{:}\mathbb{1}A) \qquad x{:}\mathbb{1}A \vdash x : A}{x{:}!A \vdash x : A}$$

Example 2: Strictness typing

- $P = \{\mathbb{L}, \mathbb{S}\}$ with $\mathbb{L} = \mathbb{N}$ (lazy) and $\mathbb{S} = \mathbb{N} \setminus \{0\} = \mathbb{1}$ (strict).

pq	$\mathbb{L}$	$\mathbb{S}$
$\mathbb{L}$	$\mathbb{L}$	$\mathbb{L}$
$\mathbb{S}$	$\mathbb{L}$	$\mathbb{S}$

$p+q$	$\mathbb{L}$	$\mathbb{S}$
$\mathbb{L}$	$\mathbb{L}$	$\mathbb{S}$
$\mathbb{S}$	$\mathbb{S}$	$\mathbb{S}$

- $p_0 = \mathbb{L} \leq \mathbb{S}$: weakening only with lazy hypotheses!

$$(x : \mathbb{L}A) \leq (x : \mathbb{S}\top) = \varepsilon$$

- Hypothesis rule usable for $\mathbb{L}$ via *sub*:

$$\frac{(x{:}\mathbb{L}A) \leq (x{:}\mathbb{S}A) \qquad x{:}\mathbb{S}A \vdash x : A}{x{:}\mathbb{L}A \vdash x : A}$$

Example 3: Relevance typing

- $P = \{\mathbb{0}, \mathbb{1}\}$ with $\mathbb{0} = \{0\}$ (unused) and $\mathbb{1} = \mathbb{N} \setminus \{0\} = \mathbb{1}$ (used).
- P is discrete: $\mathbb{0} \not\leq \mathbb{1} \not\leq \mathbb{0}$.

pq	$\mathbb{0}$	$\mathbb{1}$
$\mathbb{0}$	$\mathbb{0}$	$\mathbb{0}$
$\mathbb{1}$	$\mathbb{0}$	$\mathbb{1}$

$p+q$	$\mathbb{0}$	$\mathbb{1}$
$\mathbb{0}$	$\mathbb{0}$	$\mathbb{1}$
$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$

- $p_0 = \mathbb{0}$: weakening only with unused hypotheses!

$$(x : \mathbb{0}A) \leq (x : \mathbb{1}\top) = \varepsilon$$

- Irrelevant hypotheses are unusable

$$x{:}\mathbb{0}A \nvdash x : A$$

Variance / Monotonicity / Polarity / Positivity

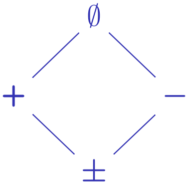
- Scala: type operator variance for subtyping.
- Coq/Agda: positivity checking for inductive/coinductive types.
- 4-point lattice:

p	variance	monotonicity	occurrence
$\emptyset$	invariant	constant	none
$+$	covariant	monotone	positive
$-$	contravariant	antitone	negative
$\pm$	mixed-variant	any function	both / don't know

- Function composition is pq.
- Combine variable occurrences in subterms with $p + q$.

Positivity Typing

- Partially ordered monoid $P = \{\emptyset, +, -, \pm\}$.



pq	$\pm$	$+$	$-$	$\emptyset$
$\pm$	$\pm$	$\pm$	$\pm$	$\emptyset$
$+$	$\pm$	$+$	$-$	$\emptyset$
$-$	$\pm$	$-$	$+$	$\emptyset$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

$p + q$	$\pm$	$+$	$-$	$\emptyset$
$\pm$	$\pm$	$\pm$	$\pm$	$\pm$
$+$	$\pm$	$+$	$\pm$	$+$
$-$	$\pm$	$\pm$	$-$	$-$
$\emptyset$	$\pm$	$+$	$-$	$\emptyset$

- Encoding: $p \leq q = p \supseteq q$ and $p + q = p \cup q$ where

$\emptyset$	$+$	$-$	$\pm$
$\{\}$	$\{+1\}$	$\{-1\}$	$\{+1, -1\}$

- Default polarity $p_0 = \emptyset$: weaken with anything.

Modal types

- Make qA first class: $A, B ::= \top \mid qA \mid A \to B$.
- Back to lambda-calculus rules plus *mod*.

$$\frac{}{x : A \vdash x : A}\ hyp \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x t : A \to B}\ abs$$

$$\frac{\Gamma \vdash t : A \to B \qquad \Delta \vdash u : A}{\Gamma + \Delta \vdash t\, u : B}\ app$$

$$\frac{\Gamma \leq \Delta \qquad \Delta \vdash t : A \qquad A \leq B}{\Gamma \vdash t : B}\ sub \qquad \frac{\Gamma \vdash t : A}{q\Gamma \vdash t : qA}\ mod$$

Linear Types, Revisited

- $P = \{\mathbb{1}, !\}$, let $\mathbb{1}A = A$.
- Observe $!!A = !A$.
- Promotion is an instance of *mod*.

$$\frac{!\Gamma \vdash t : A}{!\Gamma \vdash t : !A}\ mod$$

Conclusions & Further work

- A fairly generic simple/modal type system parametrized over a partially ordered monoid P with sum and default element p_0.
- Captures several well-known non-standard type system.
- Further work:
  - Nakano's modality for recursion.
  - Semantics?
  - Connected to Kripke models?
- Recent related work: Conor McBride uses worlds to integrate linear and dependent types.