Dialgebra-inspired Syntax for Dependent Inductive and Coinductive Types

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Outline

1. Type and Term Formation

2. Computation

Judgements

- Type B with type variables in Δ, term variables in Γ and applicable to terms $\Gamma \vdash t_i : B_i[t_1/x_1, \ldots, t_{i-1}/x_{i-1}]$:

$$\Delta \mid \Gamma \vdash B : (x_1 : A_1) \to \cdots (x_m : A_m) \to *$$

- Term t with term variables in Γ and applicable to terms $\Gamma \vdash t_i : B_i[t_1/x_1, \ldots, t_{i-1}/x_{i-1}]$:

$$\Gamma \vdash t : (x_1 : A_1) \to \cdots (x_m : A_m) \to C$$

- Judgements for contexts, which ensure strict positivity

Type Formation

$$\frac{\vdash \Delta \mid \Gamma \ \mathbf{ctx}}{\Delta, X : \overline{\Gamma}(*) \mid \emptyset \vdash X : \overline{\Gamma}(*)} \ \textbf{(TyVar-I)}$$

$$\frac{\Delta \mid \Gamma \vdash A : (x : B) \to T \qquad \Gamma \vdash t : B}{\Delta \mid \Gamma \vdash A\, t : T[t/x]} \ \textbf{(App-Ty)}$$

$$\frac{\Delta, X : \overline{\Gamma}(*) \mid \Gamma', \Gamma_k \vdash A_k : * \qquad f_k : \Gamma_k \triangleright \Gamma \qquad k = 1, \ldots, n \qquad \rho \in \{\mu, \nu\}}{\Delta \mid \Gamma' \vdash \rho(X; \overrightarrow{f}; \overrightarrow{A}) : \overline{\Gamma}(*)}$$

where

- $\Gamma = x_1 : B_1, \ldots, x_m : B_m$,
- $f_k = f_{k,1} \cdots f_{k,m}$ is a sequence of terms in context Γ_k, and
- $\overline{\Gamma}(*) = (x_1 : B_1) \to \cdots \to (x_m : B_m) \to *$.

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Type Formation

Example (Vectors)

- $\Gamma = n : \mathbb{N}$
- $\mathsf{Vec}\, A = \mu(X; (f_1, f_2); (\mathbf{1}, A \times X\, k)) : (n : \mathbb{N}) \to *$

with

$\Gamma_1 = \emptyset$ $\qquad$ $\Gamma_2 = k : \mathbb{N}$

$f_1 = (0) : \Gamma_1 \triangleright \Gamma$ $\qquad$ $f_2 = (k + 1) : \Gamma_2 \triangleright \Gamma$

$X : (n : \mathbb{N}) \to * \mid \Gamma_1 \vdash \mathbf{1} : *$ $\qquad$ $X : (n : \mathbb{N}) \to * \mid \Gamma_2 \vdash (A \times X\, k) : *$

Term formation rules I

$$\frac{\Gamma \vdash A : *}{\Gamma, x : A \vdash x : A}\ \textbf{(Proj)} \qquad \frac{\Gamma \vdash t : (x : A) \to B \qquad \Gamma \vdash s : A}{\Gamma \vdash t\, s : B[s/x]}\ \textbf{(App)}$$

$$\frac{\Gamma' \vdash \mu(X; \vec{f}; \vec{A}) : \overline{\Gamma}(*)}{\Gamma' \vdash \alpha_k : \overline{\Gamma_k}\left(A_k\{\mu(X; \vec{f}; \vec{A})/X\} \to \mu(X; \vec{f}; \vec{A})\, f_k\right)}\ \textbf{(Ind-I)}$$

$$\frac{\Gamma' \vdash \nu(X; \vec{f}; \vec{A}) : \overline{\Gamma}(*)}{\Gamma' \vdash \xi_k : \overline{\Gamma_k}\left(\nu(X; \vec{f}; \vec{A})\, f_k \to A_k\{\nu(X; \vec{f}; \vec{A})/X\}\right)}\ \textbf{(Coind-E)}$$

Term formation rules II

Let $\Gamma = x_1 : B_1, \ldots, x_m : B_m$, so that $\vec{x} : \Gamma \triangleright \Gamma$.

$$\frac{\Gamma' \vdash C : \overline{\Gamma}(*) \qquad \Gamma', \Gamma_k, y : A_k\{C/X\} \vdash g_k : (C\ f_k)}{\Gamma' \vdash \mathsf{rec}\ \vec{g} : \overline{\Gamma}(\mu(X; \vec{f}; \vec{A})\ \vec{x} \to C\ \vec{x})}\ \textbf{(Ind-E)}$$

$$\frac{\Gamma' \vdash C : \overline{\Gamma}(*) \qquad \Gamma', \Gamma_k, y : (C\ f_k) \vdash g_k : A_k\{C/X\}}{\Gamma' \vdash \mathsf{corec}\ \vec{g} : \overline{\Gamma}(C\ \vec{x} \to \nu(X; \vec{f}; \vec{A})\ \vec{x})}\ \textbf{(Coind-I)}$$

$+$ Weakening, contraction and exchange.

Example: Product type I

Example

Let $\Gamma = x_1 : C_1, \ldots, x_n : C_n$ and $\pi_A = \varepsilon$ (empty sequence)

$$\frac{\_ : * \mid \Gamma, x : A \vdash B : * \qquad \pi_A : (x : A) \triangleright \emptyset}{\Gamma \vdash \nu(\_; \pi_A; B) : *}\ \textbf{(FP-Ty)}$$

Notation: $\Pi x : A.B := \nu(\_; \pi_A; B)$

Example (λ-Abstraction)

$$\frac{\dfrac{\dfrac{\Gamma, x : A \vdash g : B}{\Gamma, x : A, \_ : \mathbf{1} \vdash g : B}\ \textbf{(Weaken)}}{\Gamma \vdash \operatorname{corec} g : \mathbf{1} \to \Pi x : A.B}\ \textbf{(Coind-I)} \qquad \langle\rangle : \mathbf{1}}{\Gamma \vdash \operatorname{corec} g\, \langle\rangle : \Pi x : A.B}\ \textbf{(App)}$$

Notation: $\lambda x.g := \operatorname{corec} g\, \langle\rangle$.

Example: Product type II

Example (Application)

Note that $B\{\Pi x : A.B/\_\} = B$.

$$\dfrac{\dfrac{\Gamma \vdash \xi : (x : A) \to (\Pi x : A.B)\,\pi_A \to B \qquad \Gamma \vdash a : A}{\Gamma \vdash \xi\, a : \Pi x : A.B \to B[a/x]}\,\textbf{(App)}}{\Gamma \vdash \xi\, a\,(\lambda x.g) : B[a/x]}\,\textbf{(App)}$$

Example (Extended Naturals)

- $\mathbb{N}^\infty = \nu(X; \varepsilon; (\mathbf{1} + X)) : *$
- No dependencies: $\Gamma = \Gamma_1 = \emptyset$
- Successor: $s_\infty = \mathsf{corec}\,\mathsf{inr} : \mathbb{N}^\infty \to \mathbb{N}^\infty$

Example (Partial Streams)

- $\Gamma = n : \mathbb{N}^\infty$
- $\mathrm{PStr}(A) = \nu(X; (s_\infty\, k, s_\infty\, k); (A, X\, k)) : (n : \mathbb{N}^\infty) \to *$
- with $\Gamma_1 = \Gamma_2 = k : \mathbb{N}^\infty$
- In a more readable style:

codata PStr (A : **Set**) : $\mathbb{N}^\infty \to$ **Set where**
  hd : (k : $\mathbb{N}^\infty$) $\to$ PStr (s_∞ k) $\to$ A
  tl : (k : $\mathbb{N}^\infty$) $\to$ PStr (s_∞ k) $\to$ PStr k

1. Type and Term Formation

2. Computation

Computation

- Define action of types on terms:

$$\frac{X_1 : \overline{\Gamma_1}(*), \ldots, X_n : \overline{\Gamma_n}(*) \mid \Gamma \vdash T : * \qquad \Gamma_i, x : A_i \vdash t_i : B_i}{\Gamma, x : F_T(\vec{A}) \vdash F_T(\vec{t}) : F_T(\vec{B})}$$

where $F_T(\vec{A}) = T\{\vec{A}/\vec{X}\}$.

Computation

- Define action of types on terms:

$$\frac{X_1 : \overline{\Gamma_1}(*), \ldots, X_n : \overline{\Gamma_n}(*) \mid \Gamma \vdash T : * \qquad \Gamma_i, x : A_i \vdash t_i : B_i}{\Gamma, x : F_T(\vec{A}) \vdash F_T(\vec{t}) : F_T(\vec{B})}$$

  where $F_T(\vec{A}) = T\{\vec{A}/\vec{X}\}$.

- Define reduction as compatible closure of

$$\mathsf{rec}\ \vec{g}\ (f_k \bullet \vec{t})\ (\alpha_k\ \vec{t}\ u) \longrightarrow g_k\left[\vec{t}/\vec{z_k}, F_{A_k}(\mathsf{rec}\ \vec{g}\ \vec{t}\ x')[u/x']/y\right]$$
$$\xi_k\ \vec{t}\ (\mathsf{corec}\ \vec{g}\ (f_k \bullet \vec{t})\ u) \longrightarrow F_{A_k}(\mathsf{corec}\ \vec{g}\ \vec{t}\ x')\left[g_k[\vec{t}/\vec{z_k}, u/y]/x'\right]$$

  Essentially from (in context Γ_k)

$$\begin{array}{ccc} C\ f_k & \xrightarrow{\mathsf{corec}\ \vec{g}\ f_k\ x'} & \nu(X; \vec{f}; \vec{A})\ f_k \\ \Big\downarrow{\scriptstyle g_k} & & \Big\downarrow{\scriptstyle \xi_k\ \vec{z}_k} \\ A_k\{C/X\} & \xrightarrow{F_{A_k}(\mathsf{corec}\ \vec{g}\ f_k\ x')} & A_k\{\nu(X; \vec{f}; \vec{A})/X\} \end{array}$$

Example (β-reduction)

$$
\begin{aligned}
\xi\, a\, (\lambda x.g) &= \xi\, a\, (\mathsf{corec}\, g\, \langle\rangle) \\
&\longrightarrow F_B(\mathsf{corec}\, g\, x')[g[a/x, \langle\rangle/\_]/x'] \\
&= x'[g[a/x, \langle\rangle/\_]/x'] && \mathsf{fv}(B) = \emptyset \\
&= g[a/x]
\end{aligned}
$$

Conclusion etc.

- Subject reduction follows from correct typing of F_T
- How about confluence, progress and strong normalisation?
- Crucial weakening of Agda's type system: Destructors cannot refer to each other. Instead, sum types are implemented as

  data Σ (A : **Set**) (B : A → **Set**) : **Set where**
  in : (a : A) → B a → Σ A B

  and we need dependent recursion to get strong elimination.
- Otherwise, the type system treats inductive and coinductive types on a par, with immediate semantics over data type complete categories.

Thank you very much
for your attention.