Typed realizability for first-order classical analysis

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Introduction

Realizability

Computational interpretation of proofs:

- In arithmetic
- In set theory
- With the excluded middle
- With the axiom of choice/comprehension scheme
- . . .

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formulas = types	formulas = specifications
proofs = programs	proofs $\subseteq$ realizers
language *defined* by the logic	language *compatible* with the logic

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language *defined* by the logic	language *compatible* with the logic

- One may add axioms (as soon as they are realized) and keep the language
- One may change the language (as soon as it realizes the logic)

...or take a model

The axiom of choice

$(A \vee \neg A) + AC$ proves the existence of non-computable functions

$$A \vee \neg A \vdash \forall n \, \exists y \, (y = 0 \Leftrightarrow A(n))$$
$$(A \vee \neg A) + AC \vdash \exists f \, \forall n \, (f(n) = 0 \Leftrightarrow A(n))$$

and take $A \equiv$ "the nth Turing machine halts on input n"

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Usual interpretation

$$PA + AC \xrightarrow{\neg\neg\ trans.} HA + AC + DNS \quad (DNS \equiv \forall x \neg\neg A \Rightarrow \neg\neg\forall x\, A)$$

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The direct method

system $T + call/cc + barrec$ realizes $PA + AC$

Extraction

From a proof of $\forall x \, \exists y \, P(x, y)$ in some theory, extract an algorithm $\mathcal{A}$ s.t.:

$$\text{for any} \quad n \in \mathbb{N} \qquad P(n, \mathcal{A}(n)) \quad \text{holds}$$

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Possible because:

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In classical analysis

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$$PA + AC \vdash \forall x \, \exists y \, P(x, y) \quad \text{implies} \quad HA + AC + DNS \vdash \forall x \, \exists y \, P(x, y)$$

and bar recursion realizes DNS: $\forall x \, \neg\neg A \Rightarrow \neg\neg\forall x \, A$ (Spector)

We use a direct method $\Rightarrow$ no CPS translation

A realizability model for classical analysis

Classical analysis: CA^{ω}

- Terms: Gödel's system T

$$t^{\sigma}, u^{\tau} \quad ::= \quad x^{\sigma} \mid (t^{\sigma\to\tau}\, u^{\sigma})^{\tau} \mid \mathsf{s}^{\cdots} \mid \mathsf{k}^{\cdots} \mid 0^{\iota} \mid \mathsf{S}^{\iota\to\iota} \mid \mathsf{rec}^{\cdots}$$

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- Formulas: negative predicate logic with inequality at all types

$$A, B \quad ::= \quad t^\sigma \neq_\sigma u^\sigma \mid \bot \mid A \Rightarrow B \mid A \wedge B \mid \forall x^\sigma\, A$$

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- Axioms:
  - Equality: reflexivity + Leibniz scheme
  - Definitions: $\forall x^\sigma \forall y^\tau\, (\mathsf{k}\, x\, y =_\sigma x)$ and similarly for s and rec
  - Arithmetic: $\forall x^\iota\, (\mathsf{S}\, x \neq_\iota 0)$ + induction scheme
  - Dependent choice:

$$\forall \vec{u}^{\vec{\tau}}\, (\forall x^\iota\, \forall y^\sigma\, \exists z^\sigma\, A\{x, y, z\} \;\Rightarrow\; \exists v^{\iota\to\sigma}\, \forall x^\iota\, A\{x, v\, x, v\,(\mathsf{S}\, x)\})$$

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- Sequents: $\Gamma \vdash A \mid \Delta$ (interpreted as $\bigwedge \Gamma \;\Rightarrow\; A \vee \bigvee \Delta$)
- Rules: introduction/elimination on the main formula. In particular:

$$\frac{\Gamma \vdash A \mid A, \Delta}{\Gamma \vdash \bot \mid A, \Delta} \quad \text{and} \quad \frac{\Gamma \vdash \bot \mid A, \Delta}{\Gamma \vdash A \mid \Delta}$$

PCF with control: μPCF

- Types: simple types with product and empty types

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- Terms: $\lambda\mu$-calculus + constants of PCF (with general fixpoint Y)

$$M, N \quad ::= \quad x \mid \lambda x.M \mid M\,N \mid \langle M, N\rangle \mid \pi_i\,M \mid [\alpha]\;M \mid \mu\alpha.M$$
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- Translation from CA^ω to μPCF:
  - Formulas to types:

$$(t \neq u)^* = \bot^* = 0 \qquad (A \wedge B)^* = A^* \times B^*$$
$$(A \Rightarrow B)^* = A^* \to B^* \qquad (\forall x\,A)^* = A^*$$

  - Proofs to typing derivations:

$$\frac{\mathfrak{p}}{\Gamma \vdash A \mid \Delta} \quad \rightsquigarrow \quad \vec{x} : \Gamma^* \vdash \mathfrak{p}^* : A^* \mid \vec{\alpha} : \Delta^*$$

+ interpretation of the axioms

Categories of continuations

Universal models of call-by-name $\lambda\mu$-calculus

- $\mathcal{C}$ distributive category
- $\mathbf{R}$ object of $\mathcal{C}$ s.t. every $\mathbf{R}^{\mathbf{X}}$ exists
- $\mathbf{R}^{\mathcal{C}}$ category of continuations
- $\mathbf{X} \parr \mathbf{Y} = \mathbf{R}^{\mathbf{X} \times \mathbf{Y}}$ *binoidal* functor

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Interpreting μPCF in $\mathbf{R}^{\mathcal{C}}$:

- interpretation of types : $\llbracket T \rrbracket \in Ob(\mathcal{C}) \qquad [T] = \mathbf{R}^{\llbracket T \rrbracket} \in Ob\left(\mathbf{R}^{\mathcal{C}}\right)$
- interpretation of typed terms:

$$[x : T_1, y : T_2 \vdash M : U \mid \alpha : V_1, \beta : V_2] \in \mathbf{R}^{\mathcal{C}}\left([T_1] \times [T_2], [U] \parr [V_1] \parr [V_2]\right)$$

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Interpretation of CA^{ω} in $\mathbf{R}^{\mathcal{C}}$

$$CA^{\omega} \xrightarrow{\_^*} \mu\text{PCF} \xrightarrow{[\_]} \mathbf{R}^{\mathcal{C}}$$

Fix now some continuation model $\mathbf{R}^{\mathcal{C}}$ of μPCF
Use μPCF syntax to describe/manipulate morphisms (drop the brackets)

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Truth values (sets of realizers): $|A| \subseteq \mathbf{R}^{\mathcal{C}}(\mathbf{1}, [A^*])$

$⫫ \subseteq \mathbf{R}^{\mathcal{C}}(\mathbf{1}, \mathbf{R})$ parameter of the model

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$$\text{ж} \perp \phi \quad \text{iff} \quad [\text{ж}]\ \phi \in \perp\!\!\!\perp \qquad ([\text{ж}]\ \phi = \mathbf{pair}(\phi, \text{ж}); \mathbf{ev})$$

Orthogonality as a property: $\phi \in |A|$ iff $\forall \text{ж} \in \|A\|, \text{ж} \perp \phi$

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Adequacy for 1st-order logic

$$\left[\left(\frac{\mathfrak{p}}{\vdash A}\right)^*\right] \in |A|$$

Interpreting the axioms

Adequacy for the axioms

$$\lambda x.x \in |a = b| \text{ if } a = b \text{ in } \mathcal{M} \qquad \mathsf{Y}\,(\lambda x.x) \in |\forall x^{\iota}\,(\mathsf{S}\,x \neq_{\iota} 0)|$$

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Realizer of induction should be of type $A^* \to (A^* \to A^*) \to A^*$

how do we know how many times to call the second argument?

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- Add axioms $(\!|\mathsf{k}|\!)$, $(\!|\mathsf{s}|\!)$, …, which are realized: $\lambda xy.x \in |(\!|\mathsf{k}|\!)|$, …

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- Add axioms $(\!|\mathsf{k}|\!)$, $(\!|\mathsf{s}|\!)$, ..., which are realized: $\lambda xy.x \in |(\!|\mathsf{k}|\!)|$, ...
- Realizer of induction:

$$\mathrm{rec} \in \left|\forall \vec{y}^{\vec{\sigma}}\,(A\{0/x\} \Rightarrow \forall^{\mathrm{r}} x^{\iota}\,(A \Rightarrow A\{\mathsf{S}\,x/x\}) \Rightarrow \forall^{\mathrm{r}} x^{\iota} A)\right|$$

Polarities in proofs

Problem

There is no falsity value $\|(\!|\_|\!)\|$!

$\Rightarrow$ we lose classical reasoning...

Polarities in proofs

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There is no falsity value $\|(\!|\_|\!)\|$!

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Indeed, $(\!|\_|\!)$ is a *positive* formula

Polarities in proofs

Problem

There is no falsity value $\|\llparenthesis\_\rrparenthesis\|$!

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Indeed, $\llparenthesis\_\rrparenthesis$ is a *positive* formula

Positive and negative formulas

$$
\begin{array}{rcl}
A^-, B^- & ::= & t \neq u \mid \bot \mid A \Rightarrow B^- \mid A^- \wedge B^- \mid \forall x^\sigma\, A^- \\
A^+, B^+ & ::= & A \Rightarrow B^+ \mid A^+ \wedge B \mid A \wedge B^+ \mid \forall x^\sigma\, A^+
\end{array}
$$

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- $\left\|A^-\right\|$ is only required for negative formulas

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- $\|A^-\|$ is only required for negative formulas
- Restriction of the sequents to $\Gamma \vdash A \mid \Delta^-$ (sufficient for $CA^{\omega \mathrm{r}}$)

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Positive and negative formulas

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\begin{array}{rcl}
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$$

- $\|A^-\|$ is only required for negative formulas
- Restriction of the sequents to $\Gamma \vdash A \mid \Delta^-$ (sufficient for $CA^{\omega\mathrm{r}}$)
- The adequacy lemma goes through

The axiom of dependent choice

Classical choice

The usual route: negative/CPS translation

$$PA^{\omega} + \text{choice} \longrightarrow HA^{\omega} + \text{choice}^{\neg\neg} \longrightarrow HA^{\omega} + \text{choice} + \text{DNS}$$

and then bar recursion realizes *intuitionistically* DNS

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Our route: avoid the CPS translation

bar recursion realizes *classically* choice

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Relies on some assumptions about $\mathbf{R}^{\mathcal{C}}$:

- $\mathbf{R}^{\mathcal{C}}([\iota], [\iota])$ contains all functions from $\mathbb{N}$ to $\mathbb{N}$
  (indeed, classical choice proves existence of non-computable functions)

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Our route: avoid the CPS translation

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Relies on some assumptions about $\mathbf{R}^{\mathcal{C}}$:

- $\mathbf{R}^{\mathcal{C}}([\iota], [\iota])$ contains all functions from $\mathbb{N}$ to $\mathbb{N}$
  (indeed, classical choice proves existence of non-computable functions)
- Continuity: if $\phi \in \mathbf{R}^{\mathcal{C}}(\mathbf{1}, [(\iota \to \sigma) \to \iota])$ then there exists some $n \in \mathbb{N}$ s.t.:

$$\forall \psi, \zeta \in \mathbf{R}^{\mathcal{C}}(\mathbf{1}, [\iota \to \sigma]), \quad \left(\forall k < n,\ \psi\,\overline{\mathsf{k}} = \zeta\,\overline{\mathsf{k}}\right) \Rightarrow \phi\,\psi = \phi\,\zeta$$

Idea of the proof

$$\forall x\, \exists y\, A(x, y) \Rightarrow \exists f\, \forall x\, A(x, f(x))$$

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0	1	2	3	4	5	6	7	8	9	10	⋯

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0	1	2	3	4	5	6	7	8	9	10	$\cdots$
v_0											

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f :

0	1	2	3	4	5	6	7	8	9	10	$\cdots$
v_0	v_1										

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0	1	2	3	4	5	6	7	8	9	10	...
v_0	v_1										

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f :

0	1	2	3	4	5	6	7	8	9	10	⋯
v_0	v_1	v_2									

...

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	v_0	v_1	v_2	v_3								

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what is $f(3)$?

it is v_3

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v_0	v_1	v_2	v_3								

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what is $f(3)$?

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what is $f(1)$?

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f :

0	1	2	3	4	5	6	7	8	9	10	$\cdots$
v_0	v_1	v_2	v_3								

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v_0	v_1	v_2	v_3								

Mary, I have f and $\forall x\, A(x, f(x))$

what is $f(3)$?

it is v_3

what is $f(1)$?

it is v_1

what is $f(6)$?

Idea of the proof

$$\forall x\, \exists y\, A(x,y) \Rightarrow \exists f\, \forall x\, A(x, f(x))$$

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James?

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0	1	2	3	4	5	6	7	8	9	10	⋯
v_0	v_1	v_2	v_3	v_4	v_5	v_6					

…

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v_0	v_1	v_2	v_3	v_4	v_5	v_6					

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v_0	v_1	v_2	v_3	v_4	v_5	v_6					

Mary, I have f and $\forall x\, A(x, f(x))$

  what is $f(3)$?

it is v_3

  what is $f(1)$?

it is v_1

  what is $f(6)$?

it is v_6

  thanks, here is $\bot$ $\longleftarrow$ Mary is continuous

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thanks, here is $\bot$ $\longleftarrow$ Mary is continuous

- Sequence built by James exists in the model, even if *uncomputable*
- CPO structure (*continuity*) of the model gives termination

Extraction

The indirect method

What we want

From a proof of $PA \vdash \forall x \exists y\, P(x, y)$, extract an algorithm $\mathcal{A}$ such that:

$$\text{for any } n \in \mathbb{N},\ P(n, \mathcal{A}(n)) \text{ holds}$$

The indirect method

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Friedman's trick

	Gödel's translation $A^{\neg}$	Friedman's translation A^{F}
$P(\vec{t})$	$\neg\neg P(\vec{t})$	$\left(P(\vec{t}) \Rightarrow F\right) \Rightarrow F$
$\exists x\, A$	$\neg\neg \exists x\, A$	$(\exists x\, A \Rightarrow F) \Rightarrow F$
$A \vee B$	$\neg(\neg A \wedge \neg B)$	$((A \Rightarrow F) \wedge (B \Rightarrow F)) \Rightarrow F$

with F some fixed formula (⚠ capture of variables)

$LK \vdash A$ implies $LJ \vdash A^{F}$ and $PA \vdash A$ implies $HA \vdash A^{F}$

The indirect method

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From a proof of $PA \vdash \forall x \exists y\, P(x,y)$, extract an algorithm $\mathcal{A}$ such that:

$$\text{for any } n \in \mathbb{N},\ P(n, \mathcal{A}(n)) \text{ holds}$$

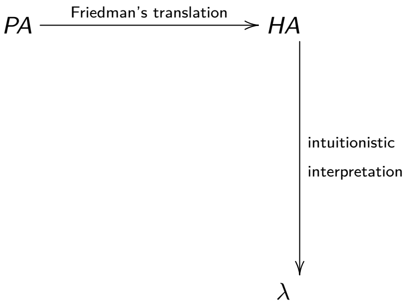
Friedman's trick

	Gödel's translation $A^{\neg}$	Friedman's translation A^{F}
$P(\vec{t})$	$\neg\neg P(\vec{t})$	$(P(\vec{t}) \Rightarrow F) \Rightarrow F$
$\exists x\, A$	$\neg\neg \exists x\, A$	$(\exists x\, A \Rightarrow F) \Rightarrow F$
$A \vee B$	$\neg(\neg A \wedge \neg B)$	$((A \Rightarrow F) \wedge (B \Rightarrow F)) \Rightarrow F$

with F some fixed formula (⚠ capture of variables)

$LK \vdash A$ implies $LJ \vdash A^F$ and $PA \vdash A$ implies $HA \vdash A^F$

$$\begin{aligned} PA \vdash \forall x \exists y\, P(x,y) &\rightsquigarrow HA \vdash (\exists y\, P(x,y) \Rightarrow F) \Rightarrow F \\ &\rightsquigarrow HA \vdash (\exists y\, P(x,y) \Rightarrow \exists y\, P(x,y)) \Rightarrow \exists y\, P(x,y) \\ &\rightsquigarrow HA \vdash \exists y\, P(x,y)\text{, hence the extracted program } M \end{aligned}$$


PA
Friedman's translation
HA
intuitionistic
interpretation
λ

The direct method

Computational content of classical logic

- Griffin:
  simply-typed λ-calculus + `call/cc` $\longleftrightarrow$ propositional classical logic
  (Curry-style)
- Parigot:
  $\lambda\mu$-calculus $\longleftrightarrow$ propositional classical logic
  (Howard-style)

  Can be extended to:
  ($\lambda\mu$-calculus + Gödel's system T) realizes *PA*

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Extraction

$$PA \vdash \forall x \exists y \neg P(x, y) \qquad \rightsquigarrow \qquad \phi(n) \in |\neg \forall^{\mathrm{r}} y\, P(n, y)|$$

now define:

$$\psi \in |\bot| \qquad \text{iff} \qquad \psi \text{ is some } m \text{ such that } \neg P(n, m) \text{ holds}$$

therefore $\lambda z.z \in |\forall^{\mathrm{r}} y\, P(n, y)|$ and $\neg P(n, \phi(n)(\lambda z.z))$ holds

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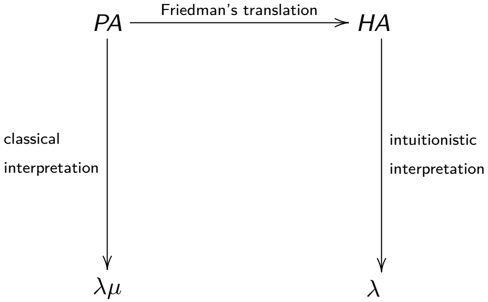
relies on taking natural numbers as realizers of $\bot$, i.e. $[\bot^*] = \mathbf{R}$ is $\mathbb{N}$
very similar to the indirect case...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CPS translation

translation from $\lambda\mu$-calculus to λ-calculus

$\lambda\mu$		λ
$x : T \vdash M : U \mid \alpha : V$	$\rightsquigarrow$	$\widetilde{x} : \widetilde{T} \to R, \widetilde{\alpha} : \widetilde{V} \vdash \widetilde{M} : \widetilde{U} \to R$

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In particular $\widetilde{0} = 1$ so: $\quad M : 0 \quad \leadsto \quad \widetilde{M} : R$

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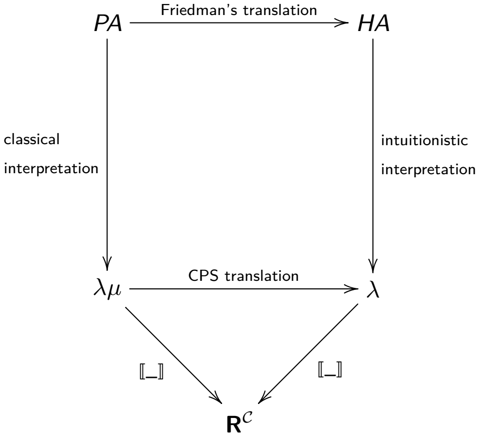
Correspondence of denotational semantics in categories of continuations

$$\begin{array}{ccc} x : T \vdash M : U \mid \alpha : V & \xrightarrow{\text{CPS}} & \widetilde{x} : \widetilde{T} \to R, \widetilde{\alpha} : \widetilde{V} \vdash \widetilde{M} : \widetilde{U} \to R \\ \downarrow [\![\_]\!] & & \downarrow [\![\_]\!] \\ \mathbf{R}^{[\![T]\!]} \to \mathbf{R}^{[\![U]\!] \times [\![V]\!]} & & [\![R]\!]^{[\![\widetilde{T}]\!]} \times [\![\widetilde{V}]\!] \to [\![R]\!]^{[\![\widetilde{U}]\!]} \end{array}$$

equal (up to currying) by choosing $[\![R]\!] = \mathbf{R}$ and $[\![\widetilde{T}]\!] = [\![T]\!]$



Didn't we gain anything?

Indirect interpretation and (naive) direct interpretation

In the end, the extracted programs are the same because:

direct	indirect
$\phi \in \lvert\bot\rvert$ iff ϕ is some m...	$F = \exists y\, P(n, y)$

are essentially the same, it amounts to:

$$[\![F^*]\!] = [\![R]\!]^{\widetilde{[\![I]\!]}} = \mathbf{R}^{[\![I]\!]} = [\bot^*] = \mathbf{R}^{[\![0]\!]} = \mathbf{R}^{\mathbf{1}} = \mathbf{R} \text{ is } \mathbb{N}$$

(same interpretation for atomic formulas and natural numbers)

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let's try to be a bit more clever...

A new direct interpretation

A new, simpler interpretation

Before:

$$\vec{A} \vdash B \mid \vec{C} \qquad \rightsquigarrow \qquad \vec{x} : \vec{A^*} \vdash M : B^* \mid \vec{\alpha} : \vec{C^*}$$

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where κ is used only for extraction

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- New parameter: $\perp\!\!\!\perp \subseteq \mathbf{R}^{\mathcal{C}}(\mathbf{1}, [I])$
- New orthogonality relation: $ж \perp \phi$ iff $\mu\kappa.\, [ж]\, \phi \in \perp\!\!\!\perp$
- And choose: $\mathbf{R}^{[\![I]\!]}$ type of natural numbers $\mathbb{N}$

  $\mathbf{R}^{[\![\perp^*]\!]} = \mathbf{R}$ empty type $\emptyset$

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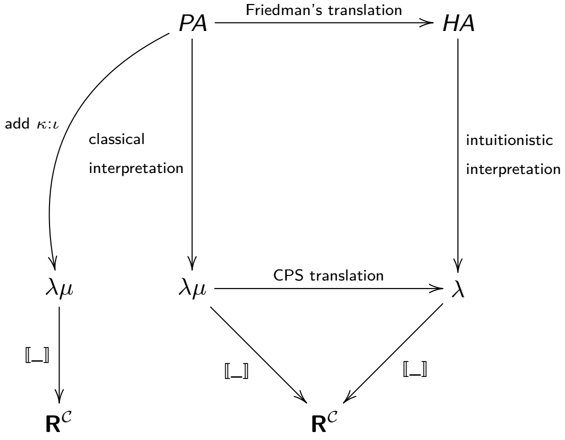
A simpler interpretation

Take the identity proof of $\perp \Rightarrow \perp$

before	now
interpreted in $\mathbb{N}^{\mathbb{N}}$	interpreted in $\left(\emptyset^{\emptyset}\right) + \mathbb{N}$

even more true at higher types



$\mathbf{R} = \emptyset, \mathbf{R}^{[\![\iota]\!]} = \mathbb{N}$ $\quad\quad$ $\mathbf{R}^{[\![\iota]\!]} = \mathbf{R} = \mathbb{N}$

Be careful

We chose:

- $\mathbf{R}^{\llbracket I \rrbracket}$ type of natural numbers $\mathbb{N}$
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- In Scott domains, $\emptyset^X$ is always $\emptyset$, it can't be $\mathbb{N}$
- But it works very well in the unbracketed games model, with **R** being the one-move arena
- Other models? (sequential algorithms, coherence spaces...)