Quotienting the Delay Monad by Weak Bisimilarity

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Delay Monad

- The delay datatype introduced by Capretta as a means to incorporate general recursion to Martin-Löf Type Theory.
- Used in this setting for modeling non-terminating behaviours.
- For a given type X, each element of $\mathrm{D}\,X$ is a possibly infinite computation that returns a value of X, if it terminates. We define $\mathrm{D}\,X$ as a coinductive type by the rules

$$\frac{}{\mathsf{now}\,x : \mathrm{D}\,X} \qquad \frac{c : \mathrm{D}\,X}{\mathsf{later}\,c : \mathrm{D}\,X}$$

- This datatype is a strong monad: unit is now and multiplication μ is "concatenation" of laters.

Weak bisimilarity

- Weak bisimilarity is defined in terms of convergence. This binary relation between $\mathrm{D}\,X$ and X relates a terminating computation to its value and is inductively defined by the rules

$$\frac{}{\mathsf{now}\, x \downarrow x} \qquad \frac{c \downarrow x}{\mathsf{later}\, c \downarrow x}$$

- Two computations are weakly R-bisimilar if they differ by a finite number of application of the constructor later, i.e., they either converge to R-related values or diverge. Weak R-bisimilarity is defined coinductively by the rules

$$\frac{c_1 \downarrow x_1 \quad x_1 R x_2 \quad c_2 \downarrow x_2}{c_1 \approx_R c_2} \qquad \frac{c_1 \approx_R c_2}{\mathsf{later}\, c_1 \approx_R \mathsf{later}\, c_2}$$

- Weak $=$-bisimilarity will simply be called weak bisimilarity and denoted $\approx$.

Quotienting by weak bisimilarity: the setoid approach

- One can consider the setoid $\hat{\mathrm{D}}(X, R) := (\mathrm{D}\,X, \approx_R)$, as in Capretta's representation of general recursive functions in TT and Benton et al.'s formalisation of domain theory in Coq (lifting of CPOs).
- The functor $\hat{\mathrm{D}}$: **Setoid** $\rightarrow$ **Setoid** is a strong monad.
- What if one considers quotients as sets? Is quotienting by weak bisimilarity preserving the monad structure in this case?

The Type Theory under consideration

- Martin-Löf Type Theory.
- (Definitional equality $:=$).
- Inductive and coinductive types.
- Uniqueness of identity proofs.
- Principle of function extensionality.
- Equality of strongly bisimilar coinductive data.
- Strong R-bisimilarity on $\mathrm{D}\,X$ is defined coinductively by the rules

$$\frac{x_1 \, R x_2}{\mathrm{now}\, x_1 (\mathrm{D}\, R) \mathrm{now}\, x_2} \qquad \frac{c_1 (\mathrm{D}\, R) c_2}{\mathrm{later}\, c_1 (\mathrm{D}\, R) \mathrm{later}\, c_2}$$

- ($\mathrm{D}\, R$ functorial lifting of R).

Adding quotient types to TT

- Quotients as particular inductive types, inspired by quotient types in Hofmann's Ph.D. thesis.
- Let X be a type and R an equivalence relation on X.
- We say thet a map $f : X \to Y$ is R-compatible if $x_1 R x_2 \to f\, x_1 = f\, x_2$. The definition extends to dependent maps (using the R-compatibility of the constructor $[-]$ introduced below).
- We postulate the existence of:
    - a carrier type X/R;
    - a R-compatible constructor $[-] : X \to X/R$;
    - a dependent eliminator: for every family $Y : X/R \to \mathcal{U}_k$ and R-compatible map $f : \prod_{x:X} Y\,[x]$ there exists a map $\mathsf{lift}\, f : \prod_{q:X/R} Y\, q$ such that for all $x : X$, $\mathsf{lift}\, f\,[x] = f(x)$.
- Propositional truncation $\|X\|$ is defined as the quotient of X by the total relation.

Weakly effective quotients

- We postulate that logical equivalence implies propositional equality for propositions (univalence for (-1)-types), i.e. for all types X, Y

$$\text{bi-imp} : \text{isProp}(X) \to \text{isProp}(Y) \to (X \leftrightarrow Y) \to X = Y$$

- Using bi-imp one can prove that quotients are weakly effective, i.e. given a type X and equivalence relation R on X the following type is inhabited:

$$\prod_{x_1, x_2 : X} ([x_1] = [x_2] \to \| x_1 R x_2 \|).$$

- (The proof requires large elimination for quotient types).

Multiplication for $\mathsf{D} - /\!\approx$

- The unit is $[-] \circ$ now.
- We are searching for a map $\mu^{\mathsf{D}} : \mathsf{D}\,(\mathsf{D}\,X/\!\approx)/\!\approx\, \to \mathsf{D}\,X/\!\approx$.
- More specifically we are searching for a $\approx$-compatible map $\mu^{\mathsf{D}\prime} : \mathsf{D}\,(\mathsf{D}\,X/\!\approx) \to \mathsf{D}\,X/\!\approx$ so that $\mu := \mathrm{lift}\,\mu^{\mathsf{D}\prime}$.
- How to construct such $\mu^{\mathsf{D}\prime}$?
- It would be nice if $\mathsf{D}\,(\mathsf{D}\,X/\!\approx)$ were a quotient of $\mathsf{D}\,(\mathsf{D}\,X)$, then we could define $\mu^{\mathsf{D}\prime} := \mathrm{lift}\,([-] \circ \mu)$.

Multiplication for $\mathsf{D} - /\approx$ (cont'd)

- But $\mathsf{D}\,(\mathsf{D}\,X/\approx)$ is not a quotient of $\mathsf{D}\,(\mathsf{D}\,X)$.
- Note that there is a map $\theta := \mathsf{lift}\,(\mathsf{D}\,[-]) : \mathsf{D}\,(\mathsf{D}\,X)/\mathsf{D}\approx\ \to \mathsf{D}\,(\mathsf{D}\,X/\approx)$.
- In general we cannot construct a map $\psi : \mathsf{D}\,(\mathsf{D}\,X/\approx) \to \mathsf{D}\,(\mathsf{D}\,X)/\mathsf{D}\approx$.

Construction of the map ψ

- More generally, let X be a type and R an equivalence relation on X.
- Under which requirements the map $\theta := \mathsf{lift}\,(\mathsf{D}\,[-]) : \mathsf{D}\,X/\mathsf{D}\,R \to \mathsf{D}\,(X/R)$ is an isomorphism?
- For finite products, finite coproducts, well-founded finitely branching trees one can define $\psi : T\,(X/R) \to T\,X/T\,R$ inverse of the canonical θ.
- Construction of ψ problematic for non-wellfounded or non-finitely branching trees.

Construction of the map ψ for function spaces

- What about function spaces?
- We say that a function $f : X \to Y$ is surjective if $\prod_{y:Y} \|\sum_{x:X} f\,x = y\|$.
- Surjectivity of the canonical map $\theta : Y^X/R^X \to (Y/R)^X$ for all X, Y, R is logically equivalent to weak existence of a section for $[-]$ for all quotients, i.e for all X and R, we have

$$\left\| \sum_{f:X/R \to X} \prod_{q:X/R} [f\,q] = q \right\|.$$

- This in turn is logically equivalent to the full axiom of choice: for all sets X, Y and relations P between X and Y, it holds that

$$\prod_{x:X} \left\| \sum_{y:Y} P\,x\,y \right\| \to \left\| \sum_{f:X \to Y} \prod_{x:X} P\,x\,(f\,x) \right\|.$$

Aside: bind for $\mathsf{D} - /\!\approx$

- An inverse of $\theta : Y^X/R^X \to (Y/R)^X$, for all X, Y, R, would solve our problems.
- Let $\psi : (\mathsf{D}\,Y/\!\approx)^X \to (\mathsf{D}\,Y/\!\approx)^X/\!\approx^X$ be such map.
- We could define bind as:

$$\begin{aligned}&\mathsf{bind}^{\mathsf{D}} : (X \to \mathsf{D}\,Y/\!\approx) \to \mathsf{D}\,X/\!\approx\, \to \mathsf{D}\,Y/\!\approx\\ &\mathsf{bind}^{\mathsf{D}}\, f\, q := \mathsf{lift}_2\,([-] \circ \mathsf{bind})\,(\psi\, f)\, q\end{aligned}$$

Construction of the map ψ for streams

- Similarly surjectivity of $\theta : X^{\mathbb{N}}/R^{\mathbb{N}} \to (X/R)^{\mathbb{N}}$ for all X, R is logically equivalent to axiom of countable choice: for all sets X and relations P between $\mathbb{N}$ and X, it holds that

$$\prod_{n:\mathbb{N}} \left\| \sum_{x:X} P\, n\, x \right\| \to \left\| \sum_{f:\mathbb{N}\to X} \prod_{n:\mathbb{N}} P\, n\, (f\ n) \right\|.$$

- For all sets X and relations P between $\mathbb{N}$ and X, we postulate the existence of acω inhabitant of the above type.
- One can prove that acω together with bi-imp implies the definability of $\psi : (X/R)^{\mathbb{N}} \to X^{\mathbb{N}}/R^{\mathbb{N}}$ inverse of θ.
- Therefore $(X/R)^{\mathbb{N}}$ is the quotient of $X^{\mathbb{N}}$ with constructor $[f]^{\mathbb{N}}\ n := [f\ n]$. Dependent eliminator and computation rule follow from θ and ψ being reciprocal inverses.

Construction of the map ψ for streams: sketch

- First one proves that the map $[-]^{\mathbb{N}} : X^{\mathbb{N}} \to (X/R)^{\mathbb{N}}$ is surjective.

$$\begin{aligned} \mathrm{box}^{\mathbb{N}}\text{-surj}' &: \prod_{s:(X/R)^{\mathbb{N}}} \left\|\sum_{t:X^{\mathbb{N}}} \prod_{n:\mathbb{N}} ([t\, n] = s\, n)\right\| \\ \mathrm{box}^{\mathbb{N}}\text{-surj}'\, s &:= \mathrm{ac}\omega\, (\lambda n \to \mathrm{box\text{-}surj}\, (s\, n)) \end{aligned}$$

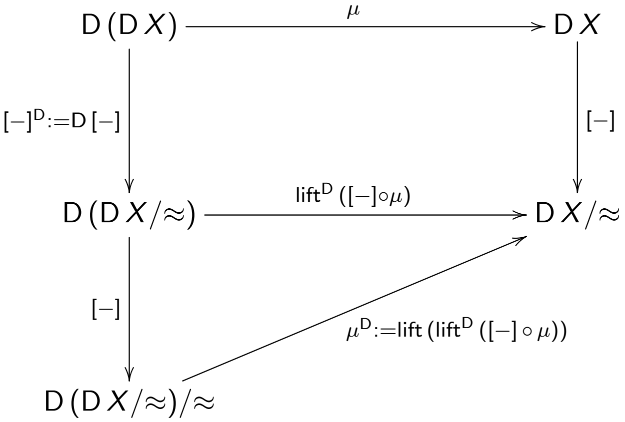
- $\mathrm{box}^{\mathbb{N}}\text{-surj} : \prod_{s:(X/R)^{\mathbb{N}}} \left\|\sum_{t:X^{\mathbb{N}}} ([t]^{\mathbb{N}} = s)\right\|$ follows using function extensionality.
- Let $s : (X/R)^{\mathbb{N}}$. There is a constant function of type $\sum_{t:X^{\mathbb{N}}} ([t]^{\mathbb{N}} = s) \to \sum_{f:(X/R)^{\mathbb{N}}} (\theta\, f = s)$. The proof of constancy uses weak effectiveness.
- Applying the induction principle of propositional truncation to the above constant function and to $\mathrm{box}^{\mathbb{N}}\text{-surj}\, s$ we get a function $\psi : (X/R)^{\mathbb{N}} \to X^{\mathbb{N}}/R^{\mathbb{N}}$ such that $\psi\, [s]^{\mathbb{N}} = [s]$ for all $s : X^{\mathbb{N}}$.

Delayed computation as streams

- Inhabitants of type $\mathsf{D}\,X$ can be represented as streams on $X + 1$ with at most one entry from X.
- There exist functions $\epsilon : \mathsf{D}\,X \to \mathbb{N} \to X + 1$ and $\pi : (\mathbb{N} \to X + 1) \to \mathsf{D}\,X$ such that $\pi\,(\epsilon\,c) = c$ for $c : \mathsf{D}\,X$.
- Using these functions one can prove that $\mathsf{D}\,(X/R)$ is the quotient of $\mathsf{D}\,X$ with constructor $[c]^{\mathsf{D}} := (\mathsf{D}\,[-])\,c$.
- We have the following recursion principle: for every type Y and $(\mathsf{D}\,R)$-compatible map $f : \mathsf{D}\,X \to Y$ there exists a map $\mathsf{lift}^{\mathsf{D}}\,f : \mathsf{D}\,(X/R) \to Y$ such that for all $c : \mathsf{D}\,X$, $\mathsf{lift}^{\mathsf{D}}\,f\,[c]^{\mathsf{D}} = f\,c$.

Multiplication is definable

- We can now define multiplication.



Conclusions and future work

- Not in this talk: compatibility of $[-] \circ \mu$ and $\mathsf{lift}^{\mathsf{D}}\,([-] \circ \mu)$ and monad laws.
- The results discussed in the talk have been formalized in Agda.
- The employment of the principle of function extensionality is unavoidable. The same seems true for uniqueness of identity proofs.
- Thorough investigation of $T\,(X/R) \cong T\,X/T\,R$ for non-wellfounded and non-finitely branching trees. How far can we go just with the principles we are postulating?