A proof with side effects of Gödel's completeness theorem suitable for semantic normalisation

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Preliminary: proving with side effects

- Classical logic seen as a side effect:
  - Direct style = a control operator (e.g. *callcc* of type Peirce's law) [Griffin 90]
  - Indirect style = continuation-passing-style/double-negation translation within intuitionistic logic
- This talk:
  - Interpreting Kripke forcing translation as indirect style for what is in direct style a monotonic memory update
  - Applying this to obtain a proof with side-effect of Gödel's completeness theorem as direct-style presentation of a proof of completeness w.r.t. Kripke semantics

Kripke forcing translation as an environment monad

Let $\geq$ be a partial order. A key clause of Kripke forcing is the interpretation of implication:

$$w \Vdash A \to B \triangleq \forall w' \geq w \, [(w' \Vdash A) \to (w' \Vdash B)]$$

The transformation

$$\Box_w A(w) \triangleq \forall w' \geq w \, A(w')$$

can be seen as a dependent environment monad, i.e. as indirect style for a monotonic memory update effect.

Direct-style for Kripke forcing

A rule for initialising the use of Kripke forcing:

$$\frac{\begin{array}{l}\Gamma, [b : x \geq t] \vdash q : T(x) \\ \Gamma \vdash r : \mathit{refl}\ \geq \\ \Gamma \vdash s : \mathit{trans}\ \geq \\ x \text{ fresh in } \Gamma \text{ and } T(t)\end{array}}{\Gamma \vdash \mathtt{set}\, x := t\, \mathtt{as}\, b/_{(r,s)}\, \mathtt{in}\, q : T(t)} \ \text{SETEFF}$$

A rule for updating:

$$\frac{\begin{array}{l}\Gamma, [b : x \geq t(x')] \vdash q : T(x) \\ \Gamma \vdash r : t(x') \geq x' \\ [x \geq u] \in \Gamma \text{ for some } u \\ x' \text{ fresh in } \Gamma\end{array}}{\Gamma \vdash \mathtt{update}\, x := t(x)\, \mathtt{of}\, x'\, \mathtt{as}\, b\, \mathtt{by}\, r\, \mathtt{in}\, q : T(t(x))} \ \text{UPDATE}$$

where we wrote T, U for $\dot{\rightarrow}$-$\dot{\forall}$-free formulas (= intuitively Σ^0_1-formulas = base types)

Gödel's completeness

Object language

We consider here the negative fragment of predicate logic as an object language (we consider $\dot{\perp}$ to be an arbitrary atom and abbreviate $\dot{\neg} A \triangleq A \dot{\rightarrow} \dot{\perp}$).

$$
\begin{array}{lll}
t & \triangleq & x \mid f(t_1, ..., t_n) \\
F, G & \triangleq & \dot{\perp} \mid \dot{P}(t_1, ..., t_n) \mid F \dot{\rightarrow} G \mid \dot{\forall} x\, F \\
\Gamma & \triangleq & \epsilon \mid \Gamma, F
\end{array}
$$

We take the following inference rules:

$$
\begin{array}{lcl}
\dot{\mathtt{Ax}}^{\Gamma,F,\Gamma'} & : & (\Gamma, F \subset \Gamma') \rightarrow (\Gamma' \vdash F) \\
\dot{\mathtt{App}}_{\rightarrow}^{\Gamma,F,G} & : & (\Gamma \vdash F \dot{\rightarrow} G) \rightarrow (\Gamma \vdash F) \rightarrow (\Gamma \vdash G) \\
\dot{\mathtt{Abs}}_{\rightarrow}^{\Gamma,F,G} & : & (\Gamma, F \vdash G) \rightarrow (\Gamma \vdash F \dot{\rightarrow} G) \\
\dot{\mathtt{Abs}}_{\forall}^{\Gamma,x,F} & : & (\Gamma \vdash F) \rightarrow (x \notin FV(\Gamma)) \rightarrow (\Gamma \vdash \dot{\forall} x\, F(x)) \\
\dot{\mathtt{App}}_{\forall}^{\Gamma,x,t,F} & : & (\Gamma \vdash \dot{\forall} x\, F) \rightarrow (\Gamma \vdash F[t/x])
\end{array}
$$

Moreover, the following is admissible:

$$
\dot{\mathtt{weak}}_{\Gamma,F}^{\Gamma'} \quad : \quad (\Gamma \subset \Gamma') \rightarrow (\Gamma \vdash F) \rightarrow (\Gamma' \vdash F)
$$

We shall also write r_F^{Γ} for a proof of $\Gamma \subset (\Gamma, F)$,

Tarskian models

A Tarskian model $\mathcal{M}$ is made of a domain $\mathcal{D}_{\mathcal{M}}$ for interpreting terms, of an interpretation of function symbols $\mathcal{F}_{\mathcal{M}}(f) : \mathcal{D}^{a_f} \rightarrow \mathcal{D}$ and of an interpretation of atoms $\mathcal{P}_{\mathcal{M}}(\dot{P}) \subset \mathcal{D}^{a_{\dot{P}}}$ (for a_f, $a_{\dot{P}}$ the arity of f, $\dot{P}$ resp.).

Truth is defined by

$$
\begin{array}{lcl}
[\![x]\!]^{\sigma}_{\mathcal{M}} & \triangleq & \sigma(x) \\
[\![f t_1 \ldots t_{a_f}]\!]^{\sigma}_{\mathcal{M}} & \triangleq & \mathcal{F}_{\mathcal{M}}(f)([\![t_1]\!]^{\sigma}_{\mathcal{M}}, \ldots, [\![t_{a_f}]\!]^{\sigma}_{\mathcal{M}}) \\
 & & \\
\models^{\sigma}_{\mathcal{M}} \dot{P}(t_1, \ldots, t_{a_{\dot{P}}}) & \triangleq & \mathcal{P}_{\mathcal{M}}(\dot{P})([\![t_1]\!]^{\sigma}_{\mathcal{M}}, \ldots, [\![t_{a_{\dot{P}}}]\!]^{\sigma}_{\mathcal{M}}) \\
\models^{\sigma}_{\mathcal{M}} \dot{\bot} & \triangleq & \mathcal{P}_{\mathcal{M}}(\dot{\bot}) \\
\models^{\sigma}_{\mathcal{M}} F \dot{\rightarrow} G & \triangleq & \models^{\sigma}_{\mathcal{M}} F \rightarrow \models^{\sigma}_{\mathcal{M}} G \\
\models^{\sigma}_{\mathcal{M}} \dot{\forall} x\, F & \triangleq & \forall t \in \mathcal{M}_D \ \models^{\sigma[x \leftarrow t]}_{\mathcal{M}} F
\end{array}
$$

Completeness w.r.t Tarskian models

Let $\mathcal{C}las$ be the theory containing $\dot{\neg}\dot{\neg}F \dot{\to} F$ for all formulas F (atoms are enough).

We define $\vdash_C F$ to be $\mathcal{C}las \vdash_M F$ in minimal logic.

A Tarskian model $\mathcal{M}$ for classical logic is a Tarskian model which satisfies $\models_{\mathcal{M}} \mathcal{C}las$ (in a classical meta-language, all Tarskian models are classical, but not in an intuitionistic meta-language).

The statement of completeness w.r.t Tarskian models for classical logic is:

$$[\forall \mathcal{M}\, \forall \sigma\, (\models^{\sigma}_{\mathcal{M}} \mathcal{C}las \to \models^{\sigma}_{\mathcal{M}} F)] \to \mathcal{C}las \vdash_M F$$

The usual proof is by contradiction, building a saturated counter-model by enumeration of the formulas.

The proof with effects we shall consider actually works for arbitrary theories, so that we shall consider instead the following statement:

$$(\forall \mathcal{M}\, \forall \sigma\ \models^{\sigma}_{\mathcal{M}} F) \to \vdash_M F$$

Completeness w.r.t. Kripke models

Kripke models

A Kripke model $\mathcal{K}$ is an increasing family of Tarskian models indexed over a set of worlds $\mathcal{W}_{\mathcal{K}}$ ordered by $\geq_{\mathcal{K}}$. In the absence of $\vee$ and $\exists$, it is enough to take $\mathcal{D}_{\mathcal{K}}$ constant.

Truth relatively to $\mathcal{K}$ at world w is defined by:

$$
\begin{array}{lcl}
[\![x]\!]^{\sigma}_{\mathcal{K}} & \triangleq & \sigma(x) \\
[\![f t_1 \ldots t_{a_f}]\!]^{\sigma}_{\mathcal{K}} & \triangleq & \mathcal{F}_{\mathcal{K}}(f)([\![t_1]\!]^{\sigma}_{\mathcal{K}}, \ldots, [\![t_{a_f}]\!]^{\sigma}_{\mathcal{K}}) \\
 & & \\
w \Vdash^{\sigma}_{\mathcal{K}} \dot{P}(t_1 \ldots t_{a_{\dot{P}}}) & \triangleq & \mathcal{P}_{\mathcal{K}}(\dot{P})_w([\![t_1]\!]^{\sigma}_{\mathcal{K}}, \ldots, [\![t_{a_{\dot{P}}}]\!]^{\sigma}_{\mathcal{K}}) \\
w \Vdash^{\sigma}_{\mathcal{K}} \dot{\bot} & \triangleq & \mathcal{P}_{\mathcal{K}}(\dot{\bot})_w \\
w \Vdash^{\sigma}_{\mathcal{K}} F \dot{\rightarrow} G & \triangleq & \forall w' \geq_{\mathcal{K}} w \, (w' \Vdash^{\sigma}_{\mathcal{K}} F \rightarrow w' \Vdash^{\sigma}_{\mathcal{K}} G) \\
w \Vdash^{\sigma}_{\mathcal{K}} \forall x \, F & \triangleq & \forall t \in \mathcal{K}_D \; w \Vdash^{\sigma[x \leftarrow t]}_{\mathcal{K}} F
\end{array}
$$

The statement of completeness w.r.t. Kripke models is:

$$(\forall \mathcal{K} \, \forall \sigma \, \forall w \in \mathcal{W}_{\mathcal{K}} \, w \Vdash^{\sigma}_{\mathcal{K}} F) \rightarrow \vdash_M F$$

Completeness w.r.t Kripke models

The "standard" proof works by building the canonical model $\mathcal{K}_0$ defined by taking $\mathcal{W}_{\mathcal{K}_0}$ to be the typing contexts ordered by inclusion, $\mathcal{D}_{\mathcal{K}_0}$ to be the terms, $\mathcal{K}_{\mathcal{F}}(f)$ to be the syntactic application of f, and $\mathcal{K}_{\mathcal{P}}(\dot{P})(\Gamma)(t_1, ..., t_{a_{\dot{P}}})$ to be $\Gamma \vdash_M \dot{P}(t_1, ..., t_{a_{\dot{P}}})$

The main lemma proves $\Gamma \vdash_M F \leftrightarrow \Gamma \Vdash_{\mathcal{K}_0} F$ by induction on F

$$
\begin{array}{llcl}
\uparrow^{\Gamma}_{F} & \Gamma \vdash_M F & \longrightarrow & \Gamma \Vdash_{\mathcal{K}_0} F \\
\uparrow^{\Gamma}_{\dot{P}(\vec{t})} & p & \triangleq & p \\
\uparrow^{\Gamma}_{F \dot{\to} G} & p & \triangleq & \Gamma' \mapsto h \mapsto m \mapsto \uparrow^{\Gamma'}_{G} \dot{\mathtt{App}}^{\Gamma',F,G}_{\to}(\dot{\mathtt{weak}}^{\Gamma'}_{\Gamma,F}(h,p), \downarrow^{\Gamma'}_{F} m) \\
\uparrow^{\Gamma}_{\dot{\forall} x\, F} & p & \triangleq & t \mapsto \uparrow^{\Gamma}_{F[t/x]} \dot{\mathtt{App}}^{\Gamma,x,F}_{\forall}(p,t)
\end{array}
$$

$$
\begin{array}{llcl}
\downarrow^{\Gamma}_{F} & \Gamma \Vdash_{\mathcal{K}_0} F & \longrightarrow & \Gamma \vdash_M F \\
\downarrow^{\Gamma}_{\dot{P}(\vec{t})} & m & \triangleq & m \\
\downarrow^{\Gamma}_{F \dot{\to} G} & m & \triangleq & \dot{\mathtt{Abs}}^{\Gamma,F,G}_{\to}(\downarrow^{\Gamma,F}_{G} (m\,(\Gamma,F)\, r^{\Gamma}_{F}\,(\uparrow^{\Gamma,F}_{F} \dot{\mathtt{Ax}}^{\Gamma_1,F,,\Gamma}(b_F)))) \\
\downarrow^{\Gamma}_{\dot{\forall} x\, F} & m & \triangleq & \dot{\mathtt{Abs}}^{\Gamma,x,F}_{\forall}(\dot{y}, \downarrow^{\Gamma}_{F[z/x]} (m\,\dot{y})) \qquad \dot{y} \text{ fresh in } \Gamma
\end{array}
$$

And finally:

$$
\mathtt{compl} \;\triangleq\; v \mapsto \downarrow^{\epsilon}_{A} (v\,\mathcal{K}_0\,\emptyset\,\epsilon) \;:\; (\forall \mathcal{K}\, \forall \sigma\, \forall w \in \mathcal{W}_{\mathcal{K}}\, w \Vdash^{\sigma}_{\mathcal{K}} F) \to \vdash_M F
$$

Completeness w.r.t. Kripke models in direct-style

Kripke forcing translation for second-order arithmetic

We consider a second-order arithmetic multi-sorted over first-order datatypes such as $\mathbb{N}$, lists, formulas, etc., and with primitive recursive atoms written $P(t_1, ..., t_{a_P})$.

$$A, B \triangleq X(t_1, ..., t_{a_X}) \mid P(t_1, ..., t_{a_P}) \mid A \wedge B \mid A \to B \mid \forall x\, A \mid \forall X\, A$$

Let $\geq$ be a partial order. We extend Kripke forcing to second order quantification.

$$\begin{array}{lcl}
w \vDash X(t_1, ..., t_{a_X}) & \triangleq & X(w, t_1, ..., t_{a_X}) \\
w \vDash P(t_1, ..., t_{a_P}) & \triangleq & P(t_1, ..., t_{a_P}) \\
w \vDash A \wedge B & \triangleq & (w \vDash A) \wedge (w \vDash B) \\
w \vDash A \to B & \triangleq & \forall w' \geq w\, [(w' \vDash A) \to (w' \vDash B)] \\
w \vDash \forall x\, A & \triangleq & \forall x\, w \vDash A \\
w \vDash \forall X\, A & \triangleq & \forall X\, (mon(X) \to w \vDash A)
\end{array}$$

where $mon(X) \triangleq \forall w\, \forall w' \geq w\, (X(w, t_1, ..., t_{a_X}) \to X(w', t_1, ..., t_{a_X}))$

Relating completeness w.r.t Tarskian models to completeness w.r.t. Kripke models

We get a stronger statement of completeness by considering completeness w.r.t Kripke models by specifically instantiating $\mathcal{W}_{\mathcal{K}}$ to be the typing contexts and $\geq$ to be inclusion of contexts.

$$(\forall(\mathcal{D}_{\mathcal{K}}, \mathcal{F}_{\mathcal{K}}, \mathcal{P}_{\mathcal{K}})\, \forall\sigma\, [\epsilon \Vdash^{\sigma}_{(\mathcal{W}_{\mathcal{K}}, \mathcal{D}_{\mathcal{K}}, \mathcal{K}_{\mathcal{F}}, \mathcal{P}_{\mathcal{K}})} F]) \rightarrow \vdash_M F$$

Now, applying forcing shows that

$$\epsilon \Vdash_x (\forall(\mathcal{D}_{\mathcal{M}}, \mathcal{F}_{\mathcal{M}}, \mathcal{P}_{\mathcal{M}})\forall\sigma \models_{(\mathcal{D}_{\mathcal{M}}, \mathcal{F}_{\mathcal{M}}, \mathcal{P}_{\mathcal{M}})} F)$$

is equivalent to

$$\forall(\mathcal{D}_{\mathcal{K}}, \mathcal{F}_{\mathcal{K}}, \mathcal{P}_{\mathcal{K}})\, \forall\sigma\, (\epsilon \Vdash_{(\mathcal{W}_{\mathcal{K}}, \mathcal{D}_{\mathcal{K}}, \mathcal{K}_{\mathcal{F}}, \mathcal{P}_{\mathcal{K}})} F)$$

and hence that *forcing over the statement of completeness w.r.t. Tarskian models is equivalent to the instantiation of the set of worlds to typing contexts of completeness w.r.t. Kripke models*

Excerpt of our meta-language with effects

$$\frac{\Gamma \vdash p : A(y) \qquad y \text{ fresh in } \Gamma}{\Gamma \vdash \lambda y.p : \forall y\, A(y)}\ \forall_I$$

$$\frac{\Gamma \vdash p : \forall x\, A(x) \qquad t \text{ updatable-variable-free or } t \text{ an updatable variable and } A(x) \text{ of type 1}}{\Gamma \vdash pt : A(t)}\ \forall_E$$

$$\frac{\Gamma \vdash p : A(X) \qquad X \text{ fresh in } \Gamma}{\Gamma \vdash p : \forall X\, A(X)}\ \forall^2_I \qquad \frac{\Gamma \vdash p : \forall X\, A(X) \qquad \Gamma \vdash q : mon_\Gamma\, B(\vec{y})}{\Gamma \vdash p : A(X)[B(\vec{y})/X(\vec{y})]}\ \forall^2_E$$

$$\frac{\Gamma, [b : x \geq t] \vdash q : T(x) \qquad \Gamma \vdash r : \mathit{refl}\ \geq \qquad \Gamma \vdash s : \mathit{trans}\ \geq \qquad x \text{ fresh in } \Gamma \text{ and } T(t)}{\Gamma \vdash \mathtt{set}\ x := t\ \mathtt{as}\ b/_{(r,s)}\ \mathtt{in}\ q : T(t)}\ \text{SETEFF}$$

$$\frac{\Gamma, [b : x \geq t(x')] \vdash q : T(x) \quad \Gamma \vdash r : t(x') \geq x' \quad [x \geq u] \in \Gamma \text{ for some } u \quad x' \text{ fresh in } \Gamma}{\Gamma \vdash \mathtt{update}\ x := t(x)\ \mathtt{of}\ x'\ \mathtt{as}\ b\ \mathtt{by}\ r\ \mathtt{in}\ q : T(t(x))}\ \text{UPDATE}$$

where C of type 1 means in the grammar $C ::= P(t_1, ..., t_{a_P}) \mid P(t_1, ..., t_{a_P}) \to C \mid \forall x\, C$ and $mon_\Gamma\, B$ means B monotonic for all updatable variables in Γ

The completeness proof in direct-style

In direct style, $\mathcal{K}_0$ is the model $\mathcal{M}_0$ defined by $\mathcal{P}_{\mathcal{M}}(\dot{P})(t_1, ..., t_{a_{\dot{P}}}) \triangleq \Gamma \vdash \dot{P}(t_1, ..., t_{a_{\dot{P}}})$ for Γ a given updatable variable

$$
\begin{array}{llcl}
\uparrow_F & \Gamma \vdash_M F & \longrightarrow & \vDash_{\mathcal{M}_0} F \\
\uparrow_{P(\vec{t})} & g & \triangleq & g \\
\uparrow_{F \dot{\to} G} & g & \triangleq & m \mapsto \uparrow_G \dot{\mathtt{App}}_{\to}^{\Gamma,F,G}(g, \downarrow_F m) \\
\uparrow_{\dot{\forall} x\, F} & g & \triangleq & t \mapsto \uparrow_{F[t/x]} \dot{\mathtt{App}}_{\forall}^{\Gamma,x,F}(g, t)
\end{array}
$$

$$
\begin{array}{llcl}
\downarrow_F & \vDash_{\mathcal{M}_0} F & \longrightarrow & \Gamma \vdash_M F \\
\downarrow_{P(\vec{t})} & m & \triangleq & m \\
\downarrow_{F \dot{\to} G} & m & \triangleq & \dot{\mathtt{Abs}}_{\to}^{\Gamma,F,G}(\mathtt{update}\, \Gamma := (\Gamma, F)\, \mathtt{of}\, \Gamma_1\, \mathtt{as}\, b_F\, \mathtt{by}\, r_F^{\Gamma}\, \mathtt{in}\, \downarrow_G (m\, (\uparrow_F \dot{\mathtt{Ax}}^{\Gamma_1,F,,\Gamma}(b_F)))) \\
\downarrow_{\dot{\forall} x\, F} & m & \triangleq & \dot{\mathtt{Abs}}_{\forall}^{\Gamma,x,F}(\dot{y}, \downarrow_{F[z/x]} (m\, \dot{y}))
\end{array}
$$

$$
\mathtt{compl} \triangleq v \mapsto \mathtt{set}\, \Gamma := \epsilon\, \mathtt{as}\, b/_{(r,s)}\, \mathtt{in} \downarrow_F^{\epsilon} (v\, \mathcal{M}_0\, \emptyset)
$$

Obviously, the resulting proof in the object language is a reification of the proof of validity as in Normalisation-by-Evaluation / semantic normalisation [C. Coquand 93, Danvy 96, Altenkirch-Hofmann 96, Okada 99, ...]

'e.

Properties of the meta-language with update effect

- Can be equipped with a reduction semantics (derived from the forcing interpretation)
- Consistent because interpretable by forcing in pure intuitionistic logic
- Inconsistent with any non-intuitionistic assumption (like classical logic)
- However, variants (under investigation) are possible:
  - Local use of classical reasoning providing Markov's principle and Double Negation Shift are possible using Ilik's variant of Kripke forcing
  - Full compatibility with classical logic using Cohen forcing to be investigated
- Preservation of consistency when mixing several uses of forcing on functional "conditions" to be investigated