# On Relating Indexed W-types with Ordinary Ones

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University of Leeds

TYPES 2015

# Correction

[4] Neil Ghani, Peter Hancock, Conor McBride, and Peter Morris. Indexed containers. Journal version (submitted for publication), July 2013.

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[4] Thorsten Altenkirch, Neil Ghani, Peter Hancock, Conor McBride, and Peter Morris. Indexed containers. Journal version (submitted for publication), July 2013.

# Extensional Type Theory: W

Formation:

$$\frac{A : \mathcal{U} \qquad B : A \to \mathcal{U}}{W_{A,B} : \mathcal{U}}$$

Introduction, elimination, and computation rules given by semantics as initial algebra

$$W_{A,B} = \mu[\![A, B]\!]$$

for polynomial endofunctor on types:

$$\begin{aligned} [\![A, B]\!] : \mathcal{U} &\to \mathcal{U} \\ X &\mapsto \sum_{a:A} X^{B(a)} \end{aligned}$$

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# Extensional Type Theory: Indexed W

Formation:

$$\frac{I : \mathcal{U} \qquad \begin{array}{c} A : \mathcal{U} \\ t : A \to I \end{array} \qquad \begin{array}{c} B : A \to \mathcal{U} \\ s : \Sigma_A B \to I \end{array}}{W_{A,B,s,t} : I \to \mathcal{U}}$$

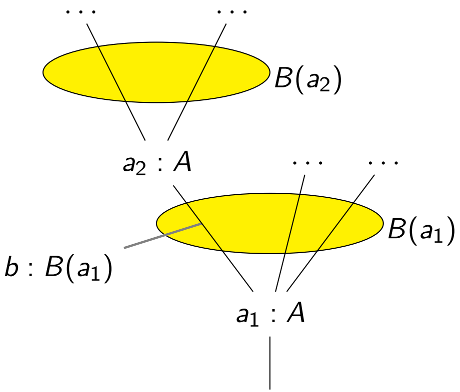
Introduction, elimination, and computation rules given by semantics as initial algebra

$$W_{A,B,s,t} = \mu[\![A, B, s, t]\!]$$

for polynomial endofunctor on *families over* $I$:

$$\begin{aligned} [\![A, B, s, t]\!] : \quad & \mathcal{U}^I \to \mathcal{U}^I \\ & (X_i)_{i:I} \mapsto \Big( \sum_{\substack{a:A, \\ t(a)=i}} \prod_{b:B(a)} X_{s(a,b)} \Big)_{i:I} \end{aligned}$$

# Intuition for W




Elements of $W_{A,B}$ are well-founded trees.

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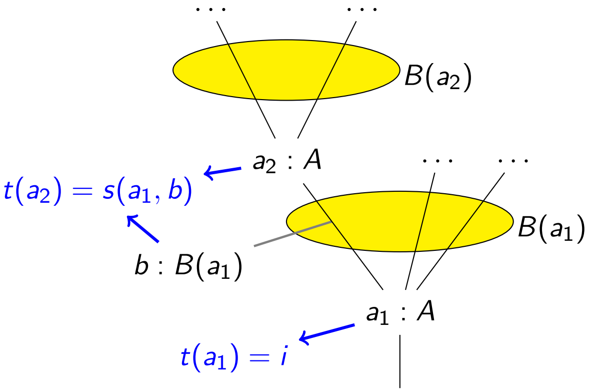


Elements of $W_{A,B,s,t}(i)$ are well-founded trees *with matching source/target $I$-annotations* (think of colours).

# Extensional Type Theory: Indexed W from W

- Idea: carve out trees $W_{A,B,s,t}$ with matching colours from "colour-untyped" trees $W_{A,B}$.

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- Idea: carve out trees $W_{A,B,s,t}$ with matching colours from "colour-untyped" trees $W_{A,B}$.
- With large elimination (not assumed): could write "colour checker" $P : W_{A,B} \to \mathcal{U}$ by recursion and define $W_{A,B,s,t} = \Sigma_{W_{A,B}} P$.
- Insight of Gambino and Hyland (TYPES 2004): can define $W_{A,B,s,t}$ using identity types (equalizers) from $W_{A,B}$ and an auxilliary type $W_{I \times A, B \circ \pi_2}$.

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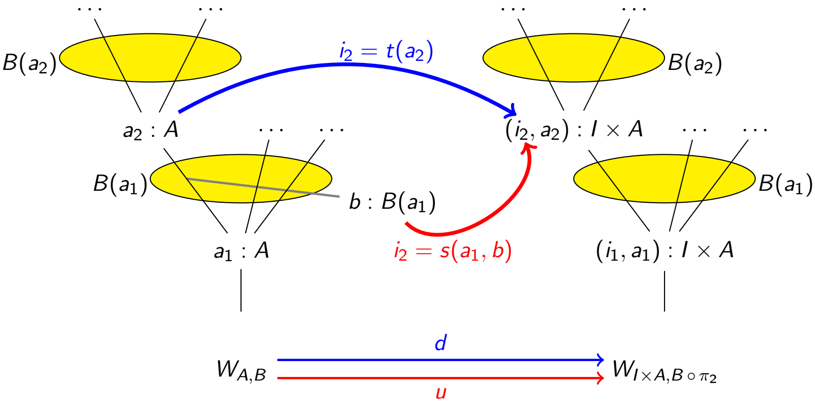

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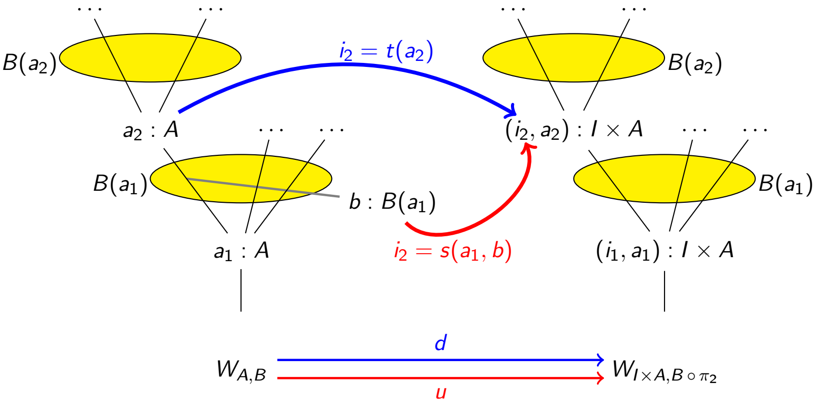


Define (total space of) $W_{A,B,s,t}$ as equalizer of $d$ and $u$:

$$W_{A,B,s,t} = \sum_{w : W_{A,B}} Id_{W_{I \times A, B \circ \pi_2}}(d(w), u(w))$$

# ITT with FunExt: Homotopy (Indexed) W

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- Homotopy indexed W-types from homotopy W-types?

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- Taking the (homotopy) equalizer introduces extraneous equalities if $A$ is not an h-set.

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- Instead, take the (homotopy) *coreflexive equalizer*

$$W_{A,B,s,t} = \sum_{\substack{w:W_{A,B},\\ p:Id(d(w),u(w))}} ap_r(p) = \alpha_w \cdot \beta_w^{-1}$$

where $\alpha : \mathrm{Id}(r \circ d, \mathrm{id})$ and $\beta : \mathrm{Id}(r \circ u, \mathrm{id})$.

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But feels kind of ad-hoc.

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(Please turn on the categorical side of your brain.)

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(Waiting. . . )

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$$\begin{aligned} \Sigma_t : \quad & \mathcal{U}^A \to \mathcal{U}^I \\ & (Z_a)_{a:A} \mapsto (\sum_{\substack{a:A \\ t(a)=i}} Z_a)_{i:I} \end{aligned}$$

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Similarly for the endofunctors underlying $W_{A,B}$ and $W_{I \times A, B \circ \pi_2}$:

$$\begin{aligned} [\![A, B, s, t]\!] &= \Sigma_t \Pi_B \Delta_s \\ [\![A, B]\!] &= \Sigma_A \Pi_B \Delta_{\Sigma_A B} \\ [\![I \times A, B \circ \pi_2]\!] &= \Sigma_{I \times A} \Pi_{B \circ \pi_2} \Delta_{I \times \Sigma_A B} \end{aligned}$$

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Can we relate these functors somehow?

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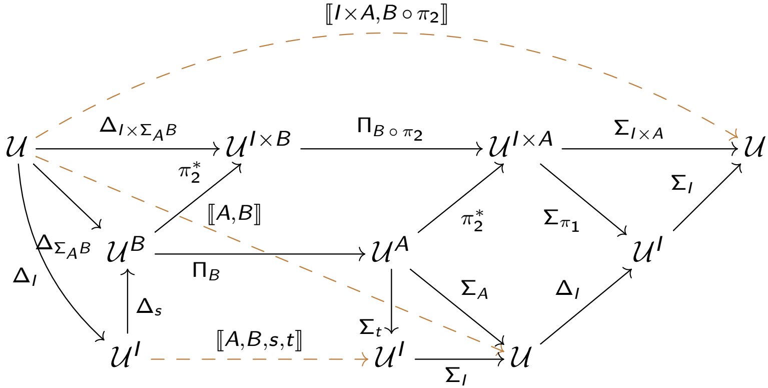

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The *Roland Rule* allows us to construct $\mu(UV)$ from $\mu(VU)$.

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Unit and multiplication form a coreflexive equalizer:

$$1 \xrightarrow{\eta} T \underset{T\eta}{\overset{\eta T}{\rightrightarrows}} T^2, \qquad T \xleftarrow{m} T^2$$

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Instead of defining total space of $W_{A,B,s,t}$ as coreflexive equalizer

$$W_{A,B,s,t} \longrightarrow W_{A,B} \underset{\color{red}{u}}{\overset{\color{blue}{d}}{\underset{\longleftarrow r \longrightarrow}{\rightrightarrows}}} W_{A,B,s,t}$$

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define $W_{A,B,s,t}$ as coreflexive equalizer

$$W_{A,B,s,t} \longrightarrow \mu(TF) \underset{\mu(T\eta F)}{\overset{\mu(\eta TF)}{\rightrightarrows}} \mu(T^2F) \qquad (\mu(mF) : \mu(T^2F) \to \mu(TF))$$

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No manual recursion needed to define analogues of $\color{blue}{d}$ and $\color{red}{u}$!

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Structure map induced by functoriality of limits:

$$
\begin{array}{ccccc}
F(W_{A,B,s,t}) & \longrightarrow & TF(\mu(TF)) & \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} & T^2F(\mu(T^2F)) \\
\alpha \big\downarrow \simeq & & \big\downarrow \simeq & & \big\downarrow \simeq \\
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Initiality of $\mu(F)$ transfers to initiality of $(W_{A,B,s,t}, \alpha)$ by an abstract fibrational argument.

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- Conjecture: Proofs about lcc quasi-categories should transfer to internal statements in ITT with FunExt (needs checking that internally defined notions agree with external ones).

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- Statements about categorical structure up to level 2 at the end of the proof require explication of categorical structure up to level $2+k$ at the beginning.
- Practically infeasible already for $k \geq 2$.

# The End

Thank you for your attention!