Set Theory in Type Theory

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How would you teach set theory to students who are familiar with type theory and proof assistants?

- Classical set theory with Zermelo-Fraenkel axioms

- Type theory with XM and impredicative Prop
- Coq as proof assistant

- Perspective very different from mathematical textbooks
- Explore an axiomatization in an expressive, explicit, and familiar logic

Axioms

$$\begin{aligned} \mathbf{S} &: \mathrm{Type} \\ \in &: \mathbf{S} \to \mathbf{S} \to \mathrm{Prop} \end{aligned}$$

$$x = y \leftrightarrow x \equiv y$$

$$\begin{aligned} z \in \emptyset &\leftrightarrow \bot \\ z \in \{x, y\} &\leftrightarrow z = x \lor z = y \\ z \in \bigcup x &\leftrightarrow \exists y \in x.\ z \in y \\ z \in \mathscr{P} x &\leftrightarrow z \subseteq x \\ z \in R @ x &\leftrightarrow \exists y \in x.\ Ryz \land \mathsf{unique}\,(Ry) \end{aligned}$$

- Replacement axiom is higher-order, $R : \mathbf{S} \to \mathbf{S} \to \mathrm{Prop}$
- Infinity and choice are not needed for this talk

Classes

- A **class** is a predicate $p : \mathbf{S} \to \mathrm{Prop}$
- Not every class can be represented as a set, e.g., $\lambda x.\ x \notin x$

- Type theory provides classes and relations on classes
- Classes are not formalized by Zermelo-Fraenkel set theory
- Von-Neumann-Gödel-Bernays set theory accommodates sets and classes in first-order logic

Separation and Description

can be expressed with replacement

$$
\begin{aligned}
z \in x \cap p &\leftrightarrow z \in x \wedge px && \text{separation} \\
p \ulcorner p \urcorner &\leftarrow p \text{ unique and inhabited} && \text{description}
\end{aligned}
$$

An operator that maps relations R on $\mathbf{S}$ to total functions $f : \mathbf{S} \to \mathbf{S}$ such that f agrees with R on unique images can be expressed (i.e., $Rx(fx)$)

Numbers and Ordered Pairs

can be represented as sets

- Functions, numbers, and pairs already exist in type theory
- Can express functions $\overline{\cdot} : \mathbf{N} \to \mathbf{S}$, $\mathrm{succ} : \mathbf{S} \to \mathbf{S}$, and $\mathrm{pred} : \mathbf{S} \to \mathbf{S}$ such that:

$$\begin{aligned} \overline{m} = \overline{m} &\leftrightarrow m = n \\ \mathrm{succ}\ \overline{n} &= \overline{n+1} \\ \mathrm{pred}\ \overline{n+1} &= \overline{n} \end{aligned}$$

- Can express functions $\mathrm{pair} : \mathbf{S} \to \mathbf{S} \to \mathbf{S}$, $\mathrm{fst} : \mathbf{S} \to \mathbf{S}$, and $\mathrm{snd} : \mathbf{S} \to \mathbf{S}$ such that:

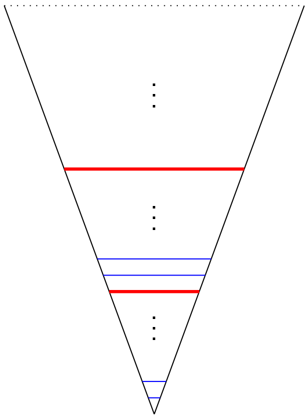
$$\begin{aligned} \mathrm{pair}\ x\ y = \mathrm{pair}\ x'\ y' &\to x = x' \wedge y = y' \\ \mathrm{fst}\ (\mathrm{pair}\ x\ y) &= x \\ \mathrm{snd}\ (\mathrm{pair}\ x\ y) &= y \end{aligned}$$

- [Barras 2010] [von Neumann 1923] [Kuratowski 1921]

Can Construct Models of Axioms

- Without infinity hereditarily finite sets suffice
- Use Ackermann encoding into numbers
- Need strong excluded middle for replacement ($\mathrm{Prop} \simeq \mathrm{bool}$)
- Aczel, Werner, Miquel construct models with infinite sets

Cumulative Hierarchy



- Horizontal lines represent stages (successors and limits)
- Blue lines also represent slices
- Every well-founded set appears in some slice
- Stages are well-ordered
- Every well-ordered set is order-isomorphic to a unique segment

Well-Founded Sets

- Define class $\mathscr{W}$ of **well-founded sets** inductively

$$\frac{x \subseteq \mathscr{W}}{x \in \mathscr{W}}$$

- Well-founded sets are defined as sets that admit ϵ-induction
- Inductive definition unknown in set theory
- Regularity axiom can be expressed as $\forall x.\, x \in \mathscr{W}$
- First-order characterization of $x \in \mathscr{W}$ seems to require infinity (to express transitive closure)
- First-order characterization of $x \in \mathscr{W} \cap \mathscr{T}$ straightforward
- Aczel [1988] studies non-well-founded sets
- $\mathscr{W}$ cannot be represented as a set

Stages of Cumulative Hierarchy

- Define class $\mathscr{Z}$ of **cumulative sets** inductively

$$\frac{x \subseteq \mathscr{Z}}{\bigcup x \in \mathscr{Z}} \qquad \frac{x \in \mathscr{Z}}{x \cup \mathscr{P}x \in \mathscr{Z}}$$

- $\mathscr{Z}$ well-ordered by $\subseteq$, unbounded, $\emptyset$ least element
- $\mathscr{W} \equiv \bigcup \mathscr{Z}$
- $x \subset y$ iff $x \in y$ for all $x, y \in \mathscr{Z}$
- $x \cup \mathscr{P}x = \mathscr{P}x$ if $x \in \mathscr{Z}$ since $\mathscr{Z} \subseteq \mathscr{T}$
- Definition of $\mathscr{Z}$ is instance of tower construction
- $\mathscr{Z}$ usually defined with transfinite induction on ordinals

Ordinals

- Define class $\mathscr{O}$ of **ordinals** inductively

$$\frac{x \subseteq \mathscr{O}}{\bigcup x \in \mathscr{O}} \qquad \frac{x \in \mathscr{O}}{x \cup \{x\} \in \mathscr{O}}$$

- Every cumulative slice contains exactly one ordinal
- Every ordinal is the set of all smaller ordinals
- Every well-ordered set is order isomorphic to a unique ordinal
- $\mathscr{O}$ order isomorphic with $\mathscr{Z}$
- Definition of $\mathscr{O}$ is instance of tower construction

First-Order Characterization of Ordinals

- Ordinals are hereditarily transitive and well-founded sets [Bernays 1931]

- $x \in \mathscr{O}$ iff $x \in \mathscr{T}$ and $x \subseteq \mathscr{T}$ and $x \in \mathscr{W}$
- $x \in \mathscr{O}$ iff $x \in \mathscr{T}$ and $x \subseteq \mathscr{T}$ and $\mathscr{P}x \subseteq \mathscr{R}$

- $\mathscr{T} := \{\, x \mid \forall y \in x.\ y \subseteq x \,\}$ transtive sets
- $\mathscr{R} := \{\, x \mid \exists y \in x\ \forall z \in x.\ z \notin y \,\}$ regular sets

- If $x \in \mathscr{T}$, then $x \in \mathscr{W}$ iff $\mathscr{P}x \subseteq \mathscr{R}$

- Corresponding inductive characterization:

$$\frac{x \in \mathscr{T} \qquad x \subseteq \mathscr{O}}{x \in \mathscr{O}}$$

Tower Construction for Sets

- Assume $f : \mathbf{S} \to \mathbf{S}$
- Define class T of sets inductively:

$$\frac{x \subseteq T}{\bigcup x \in T} \qquad \frac{x \in T}{x \cup f\,x \in T}$$

- T is well-ordered by $\subseteq$, $\emptyset$ least element
- $x \cup f\,x$ successor of x if $x \in T$ not maximal
- Every segment of T can be represented as a set
- If f preserves transitivity and well-foundedness, and $x \in f\,x$ for all x,
  - T unbounded
  - T cannot be represented as a set
  - Every well-ordered set is isomorphic to a proper segment of T
  - $x \in y$ iff $x \subset y$ for all $x, y \in T$

Tower Construction for Complete Partial Orders

- Assume type X and partial order $\leq$
- Assume $x_0 : X$
- Assume increasing function $f : X \to X$ (i.e., $x \leq f\,x$)
- Assume family $\mathscr{S}$ of classes on X, closed under subclasses
- Assume function $\bigsqcup$ that yields supremum for every $p \in \mathscr{S}$
- Define class T on X inductively:

$$\frac{}{x_0 \in T} \qquad \frac{x \in T}{f\,x \in T} \qquad \frac{p \subseteq T \qquad p \in \mathscr{S} \qquad p \text{ inhabited}}{\bigsqcup p \in T}$$

- T well-ordered by $\leq$ (x_0 least element, f yields successors)
- T unbounded iff f has no fixed point in T
- If $T \in \mathscr{S}$, then $\bigsqcup T$ is unique fixed point of f in T (Bourbaki-Witt theorem)
- See forthcoming paper at ITP 2015

Final Remarks

- Type theory provides expressive language for talking about sets and classes
  - more natural than first-order logic
  - first-order encodings are low-level and tedious; e.g.,
    - well-founded sets
    - von-Neumann-Gödel-Bernays set theory
- Many aspects of set theory can be formulated more generally at the level of type theory:
  - Well-orderings
  - Transfinite recursion
  - Tower construction
  - Well-ordering theorem
- Cumulative hierarchy can be considered before ordinals, transfinite recursion is not needed