Cubical sets as a classifying topos

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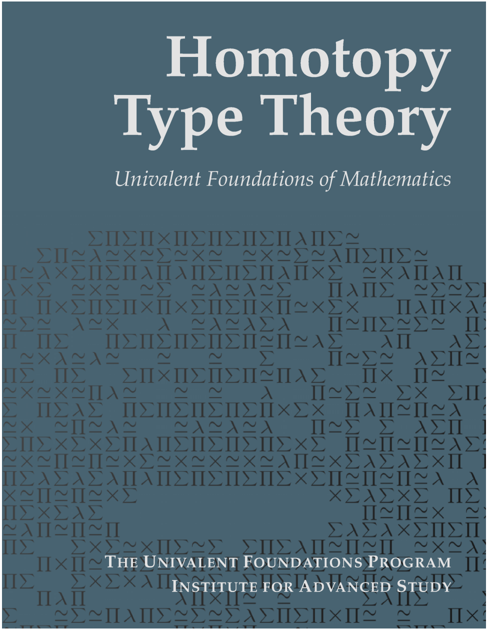

Homotopy
Type Theory
Univalent Foundations of Mathematics
The Univalent Foundations Program
Institute for Advanced Study

Homotopy type theory

Towards a new practical foundation for mathematics.
Closer to mathematical practice, inherent treatment of equivalences.

Towards a new design of proof assistants:
Proof assistant with a clear denotational semantics, guiding the addition of new features.

Concise computer proofs. (deBruijn factor < 1 !).

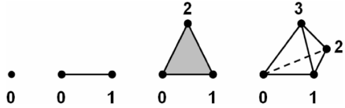
Simplicial sets

Univalence modeled in Kan fibrations of simplicial sets.
Simplicial sets are a standard example of a classifying topos.

The topos of simplicial sets models ETT. Kan fibrations are build on top of this. Voevodsky's HTS provides both fibrant and non-fibrant types.

Simplicial sets

Simplex category Δ:
finite ordinals and monotone maps
Simplicial sets $\hat{\Delta}$.



Roughly: points, equalities, equalities between equalities, ...

Cubical sets

Problem: computational interpretation of univalence and higher inductive types.
Solution (Coquand et al): Cubical sets
Cubical sets with connections, diagnonals, . . .
What does this classify?

Cubical sets

points, lines, cubes, ...
Fin= finite sets with all maps
Let T be the monad on Fin that adds two elements $0, 1$.
$Cubes = Fin_T$. Cubes with diagonals.

Interpretation: Finite sets (dimensions)
Operations: face maps (e.g. left, right end point)

Cubical sets

Coquand has more structure:
line in direction i, left endpoint $i = 0$, right endpoint $i = 1$.
$1 - i$: Symmetries
$i = j$: Diagonal of a square, cube, ...
$\wedge, \vee$: 'Connections'
De Morgan algebra: distributive lattice with $0, 1$ satisfying De Morgan laws.
Let $DM(I)$ be the free DM-alg monad on Fin.
$Cube := Fin_{DM}$ and cubical sets $\widehat{Fin_{DM}}$

Classifying category

Grandis-Mauri: classifying categories for various cubical sets.
No treatment of cubes with diagonals.
We show that the underlying cube category is the opposite of the Lawvere theory of De Morgan algebras.

Lawvere theory

Classifying categories for categories with Cartesian product.
Alternative to monads in CS (Plotkin-Power)
For algebraic (=finite product) theory T, the *Lawvere theory* $C_{fp}[T]$ is the opposite of the category of free finitely generated models.
models of T in any finite product category category E correspond to product-preserving functors $C_{fp}[T] \to E$.
Lemma: The Kleisli category Fin_{DM} is precisely the *opposite* of the Lawvere theory for DM-algebras:
maps $I \to DM(J)$ are equivalent to DM-maps $DM(I) \to DM(J)$, as $DM(I)$ is free.

Classifying topos

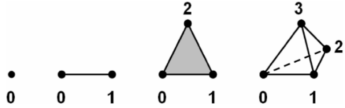
To obtain the classifying topos for an algebraic theory T, we first need to complete the Lawvere theory with finite limits, i.e. to consider the category C_{fl} as the *opposite* of finitely *presented* T-models.
Then $C_{fl}^{op} \to Set$, i.e. functors on finitely presented T-algebras, is the classifying topos. This topos contains a generic T-algebra M.
T-algebras in any topos $\mathcal{F}$ correspond to *left exact left adjoint* functors from the classifying topos to $\mathcal{F}$.

Classifying topos

Example:
Set^{Fin} classifies the Cartesian theory with one sort.
Used for variable binding (Fiore, Plotkin,Turi).
Replaces Pitts' use of nominal sets for the cubical model.
(TYPES 2014)

Simplicial sets

Standard simplices are constructed from the linear order on $\mathbb{R}$ in Set.



Can be done in any topos with a linear order.
Geometric realization becomes a geometric morphism by moving from spaces to toposes.
Equivalence of cats:

$$Orders(\mathcal{E}) \to Hom(\mathcal{E}, \hat{\Delta})$$

assigns to an order I in $\mathcal{E}$, the geometric realization defined by I.
Simplicial sets classify the *geometric* theory of strict linear orders.

Classifying topos of cubical sets

Let FG be the category of *free finitely generated* DM-algebras and let FP the category of *finitely presented* ones. We have a fully faithful functor $f : FG \to FP$. This gives a geometric morphism ϕ between the functor toposes. Since f is fully faithful, ϕ is an embedding.

Classifying topos of cubical sets

The subtopos Set^{FG} of the classifying topos for DM-algebras is given by a quotient theory, the theory of the model $\phi^* M$. This model is given by pullback and thus is equivalent to the canonical DM-algebra $\mathbb{I}(m) := m$ for each $m \in FG$.
So cubical sets are the classifying topos for 'free DM-algebras'.
Each finitely generated DM-algebra has the disjunction property and is strict, $0 \neq 1$. These properties are geometric and hence also hold for $\mathbb{I}$. This disjunction property is important in the implementation.
We have an ETT with an internal 'interval'

Kleene algebra

This result can be generalized to related algebraic structures, e.g. Kleene algebras.
A Kleene algebra is a DM-algebra with the property for all x, y,

$$x \wedge \neg x \leq y \vee \neg y$$

Kleene algebras precisely capture the lattice theory of the unit interval.
Free finitely generated Kleene algebras also have the disjunction property. Boolean algebras don't.

Categorical models of HoTT

van den Berg, Garner. Path object categories.
Usual path composition is only h-associative.
Moore paths can be arbitrary length.
category freely generated from paths of length one.
Moore paths: strict associativity, but non-strict involution.
Docherty model of MLTT in cubical sets with connections, but no diagonals.
Now: a topos with an internal 'interval'.
Apply vdB/G-D construction. However, work internally in the topos of cubical sets using the generic DM-algebra.
Simplifies computation substantially.

Related work

A model of intensional type theory.
Like Voevodsky's HTS: intensional identity types inside the extensional type theory of a topos.

Independently, Awodey showed that Cartesian cubical sets (without connections or reversions) classify strictly bipointed objects.
It is likely that much of Awodey's constructions of the cubical methods can be extended based on the interval above and would give Coquand's model.

Conclusion

- Towards a more categorical description of the cubical model
- Cubical sets as the classifying topos of the geometric theory of 'free DM-algebras'.
- Towards a proof assistant with a clear denotational semantics Cubical
- Towards elementary higher topos theory, topos theoretic methods in type theory