A Tale of Three Algorithms:
Linear Time Suffix Array Construction

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Linear time suffix array construction

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  - the problem
  - significance
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- Three algorithms from June 2003
  - description in parallel
  - differences and similarities

Suffix array construction

Sort the suffixes of a text lexicographically

- text $T = T[0, n) = t_0 t_1 \cdots t_{n-1}$
- suffix $S_i = T[i, n) = t_i t_{i+1} \cdots t_{n-1}$

Output: suffix array

- sorted array of suffixes
- suffix S_i is represented by i

0 1 2 3 4 5
banana

0	banana
1	anana
2	nana
3	ana
4	na
5	a
6	

$\Longrightarrow$

0 1 2 3 4 5
banana

6	
5	a
3	ana
1	anana
0	banana
4	na
2	nana

Applications

- Full-text indexing
  - binary and backward search
- Construction of other index structures
  - suffix tree
  - compressed indexes
- Text compression
  - Burrows-Wheeler transform
- Finding regularities
  - longest repetition, etc.
- Comparing two or more strings
  - $T = T_1 \# T_2$

0 1 2 3 4 5
banana

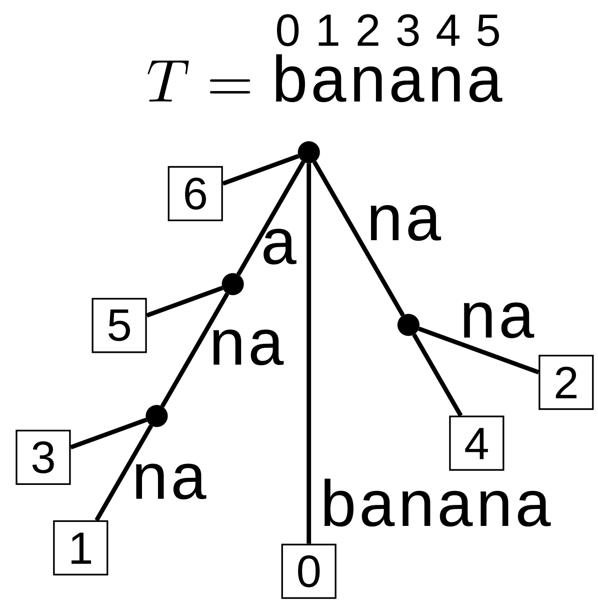
0	6	
1	5	a
2	3	ana
3	1	anana
4	0	banana
5	4	na
6	2	nana

Many of the applications need the longest common prefix array

- computable in linear time [Kasai et al., 2001]

Suffix array vs. Suffix tree

- Suffix arrays are no more an inferior simplification of suffix trees
- many recent suffix array algorithms are
  - efficient in theory and practice
  - different from suffix tree algorithms
  - nontrivial, even surprising
- case in point: linear time construction



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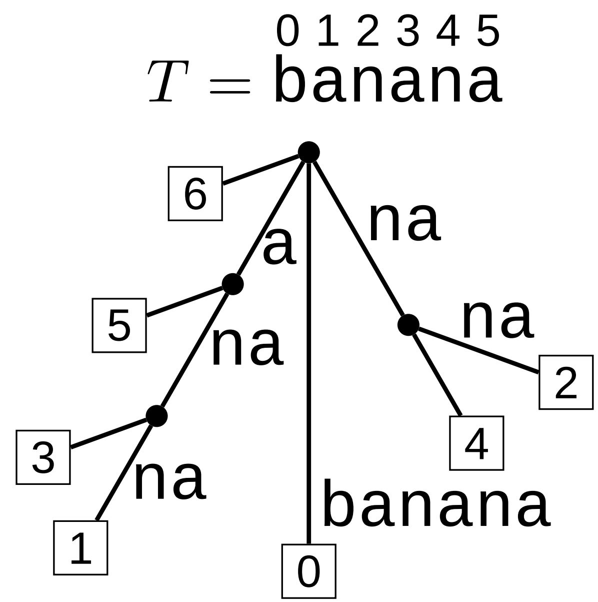


"In 2003 four papers have been published that collectively seem to establish the superiority of the suffix array over the suffix tree"

"Thus, if I were writing Chapter 5 today instead of in 2000/2001, I believe I would take a completely different approach: presenting suffix arrays as the main data structure"

— Bill Smyth: Errata on *Computing Patterns in Strings*

Alphabet

General alphabet

- only character comparisons in constant time
- lower bound $\Omega(n \log n)$ on suffix sorting

Constant alphabet

- constant number of distinct characters

Integer alphabet

- characters are integers from the range $[1, n]$

Alphabet

General alphabet

- only character comparisons in constant time
- lower bound $\Omega(n \log n)$ on suffix sorting

Constant alphabet

- constant number of distinct characters

Integer alphabet

- characters are integers from the range $[1, n]$
- order preserving renaming for other alphabets: sort characters and rename them with ranks
- linear time algorithm for integer alphabet

  $\Longrightarrow$ sorting suffixes is no harder than sorting characters

```
banana
  ⇓
 abn
  ⇓
 123
  ⇓
213131
```

6	
5	1
3	131
1	13131
0	213131
4	31
2	3131

History of linear time suffix array construction

1973 Suffix tree [Weiner]

- linear time construction for constant alphabet

1990 Suffix array [Manber & Myers]

- linear time construction only by conversion from suffix tree

1997 Integer alphabet [Farach]

- linear time suffix tree construction for integer alphabet

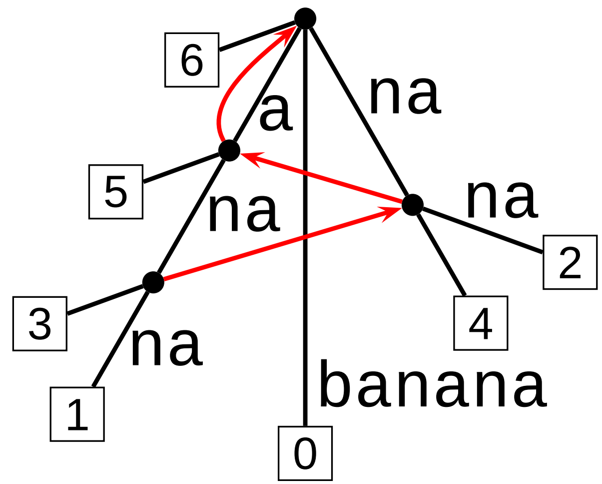
2003 Direct linear time suffix array construction [Ko & Aluru][Kim & al.][Kärkkäinen & Sanders]

- integer alphabet

Linear time suffix tree construction

- incremental algorithms
  [Weiner ’73] [McCreight ’76] [Ukkonen ’95]
  - add suffixes/characters one at a time
  - constant alphabet
  - suffix links needed
  - suffix automaton [Blumer et al., ’83]

$S =$ banana

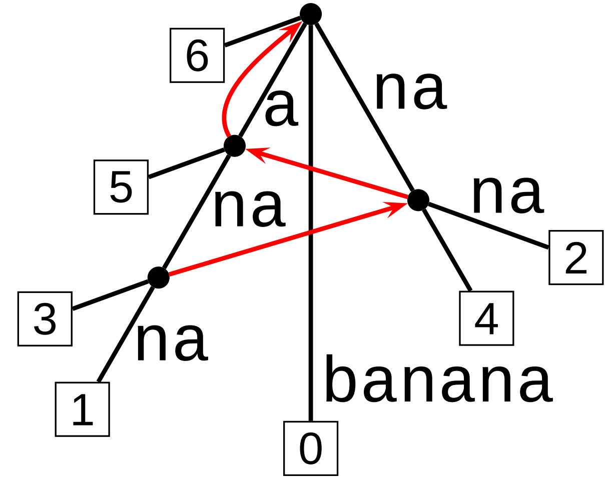


Linear time suffix tree construction

- divide-and-conquer [Farach '97]
  1. build suffix tree of $R = [t_0 t_1][t_2 t_3]\ldots$
  2. build odd and even tree
  3. merge them (complicated)

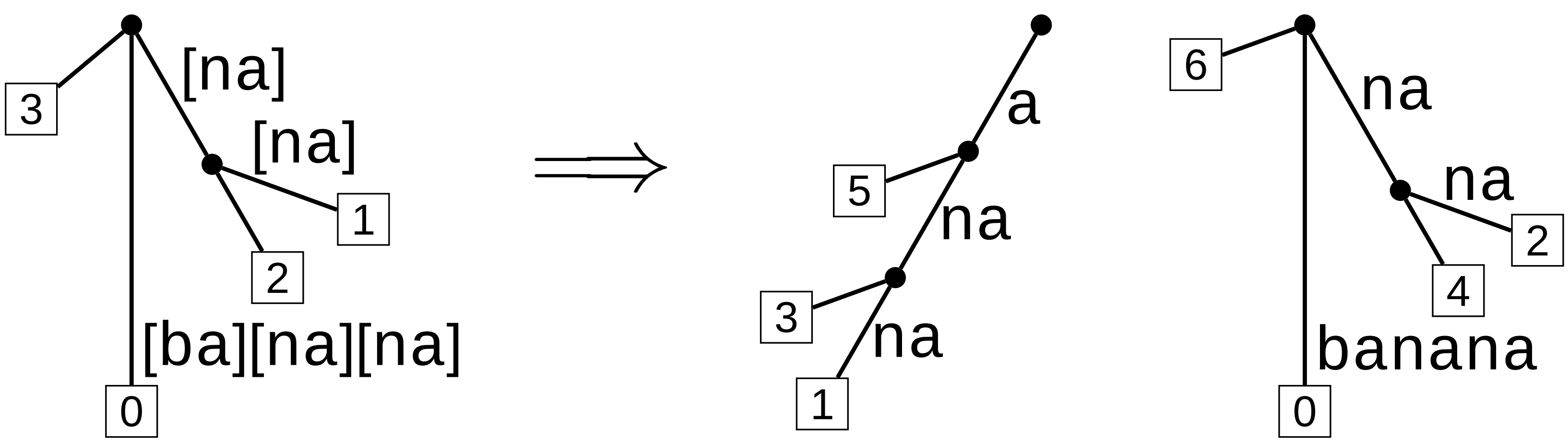
- integer alphabet
- suffix links needed in merging

$S =$ banana



$R =$ [ba][na][na]

odd tree even tree



Linear time suffix array construction

- three algorithms in June 2003

  A2: [Kim, Sim, Park & Park., CPM '03]

  A3: [Kärkkäinen & Sanders, ICALP '03]

  Ax: [Ko & Aluru, CPM '03]

- common structure: divide-and-conquer

  0. Choose a sample $\mathcal{S}$ of suffixes
  1. Sort the sample $\mathcal{S}$ by recursion
  2. Sort other suffixes $\bar{\mathcal{S}}$ using sorted $\mathcal{S}$
  3. Merge $\mathcal{S}$ and $\bar{\mathcal{S}}$

- rest of talk
  - step-by-step description

    Step 0 $\longrightarrow$ Step 3 ($\longrightarrow$ Step 1 $\longrightarrow$ Step 2)
  - all algorithms in parallel

Time complexity

0. Choose a sample $\mathcal{S}$ of suffixes
1. Sort the sample $\mathcal{S}$ by recursion
2. Sort other suffixes $\bar{\mathcal{S}}$ using sorted $\mathcal{S}$
3. Merge $\mathcal{S}$ and $\bar{\mathcal{S}}$

- integer alphabet
- excluding recursive call everything is linear
- recursion on text R over integer alphabet with $|R| = |\mathcal{S}| \leq 2n/3$
- time complexity $T(n) \leq \mathcal{O}(n) + T(2n/3) = \mathcal{O}(n)$

Step 0: Compute sample

A2: $\mathcal{S} = \{S_i \mid i \bmod 2 \neq 0\} =$ odd suffixes [Kim & al.]

- sample size $n/2$

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A3: $\mathcal{S} = \{S_i \mid i \bmod 3 \neq 0\} = \{S_1, S_2, S_4, S_5, S_7 \ldots\}$ [K & Sanders]

- sample size $2n/3$

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- sample size $2n/3$

Ax: $\mathcal{S} =$ smaller of $\{S_i \mid S_i < S_{i+1}\}$ and $\{S_i \mid S_i > S_{i+1}\}$ [Ko & Aluru]

- sample size $\leq n/2$
- w.l.o.g. assume $\mathcal{S} = \{S_i \mid S_i < S_{i+1}\}$
- $S_i \in \mathcal{S} \quad \Longleftrightarrow \quad t_i < t_{i+1}$ or $t_i = t_{i+1}$ and $S_{i+1} \in \mathcal{S}$

Step 0: Compute sample: Example

$$S = \overset{0\,1\,2\,3\,4\,5}{\text{banana}}$$

A2: $\mathcal{S} = \{S_i \mid i \bmod 2 \neq 0\}$

1	anana
3	ana
5	a

A3: $\mathcal{S} = \{S_i \mid i \bmod 3 \neq 0\}$

1	anana
2	nana
4	na
5	a

Ax: $\mathcal{S} = \{S_i \mid S_i < S_{i+1}\}$

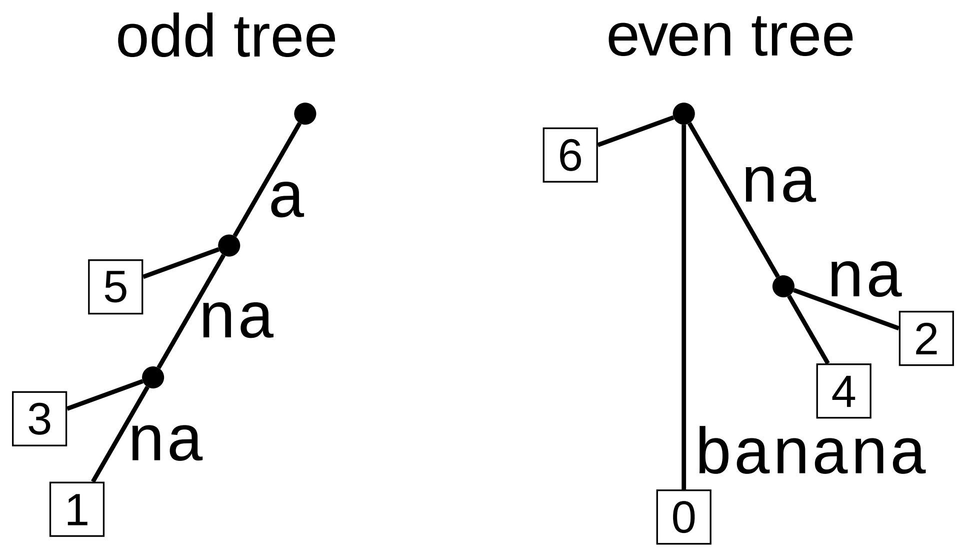
1	anana	$<$	nana
3	ana	$<$	na

banana	$>$	anana
nana	$>$	ana
na	$>$	a
a	$>$	

Step 3: Merge $\mathcal{S}$ and $\bar{\mathcal{S}}$

A2: $\mathcal{S} = \{S_i \mid i \bmod 2 \neq 0\}$ $\bar{\mathcal{S}} = \{S_j \mid j \bmod 2 = 0\}$

- very complicated (simulates suffix tree?)



Step 3: Merge $\mathcal{S}$ and $\bar{\mathcal{S}}$

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A3: $\mathcal{S} = \{S_i \mid i \bmod 3 \neq 0\}$ $\quad\quad$ $\bar{\mathcal{S}} = \{S_j \mid j \bmod 3 = 0\}$

- standard comparison-based merge
- need to compare $S_i \in \mathcal{S}$ and $S_j \in \bar{\mathcal{S}}$:
- $i \bmod 3 = 1 \quad \Longrightarrow \quad S_{i+1}, S_{j+1} \in \mathcal{S}$
  $\Longrightarrow$ compare (t_i, S_{i+1}) and (t_j, S_{j+1})
- $i \bmod 3 = 2 \quad \Longrightarrow \quad S_{i+2}, S_{j+2} \in \mathcal{S}$
  $\Longrightarrow$ compare (t_i, t_{i+1}, S_{i+2}) and (t_j, t_{j+1}, S_{j+2})

Step 3: Merge $\mathcal{S}$ and $\bar{\mathcal{S}}$

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- standard comparison-based merge
- need to compare $S_i \in \mathcal{S}$ and $S_j \in \bar{\mathcal{S}}$:
- $i \bmod 3 = 1 \quad \Longrightarrow \quad S_{i+1}, S_{j+1} \in \mathcal{S}$
  $\Longrightarrow$ compare (t_i, S_{i+1}) and (t_j, S_{j+1})
- $i \bmod 3 = 2 \quad \Longrightarrow \quad S_{i+2}, S_{j+2} \in \mathcal{S}$
  $\Longrightarrow$ compare (t_i, t_{i+1}, S_{i+2}) and (t_j, t_{j+1}, S_{j+2})

Ax: $\mathcal{S} = \{S_i \mid S_i < S_{i+1}\}$ $\quad$ $\bar{\mathcal{S}} = \{S_j \mid S_j > S_{j+1}\}$

- let $\mathcal{S}_c = \{S_i \in \mathcal{S} \mid t_i = c\}$ and $\bar{\mathcal{S}}_c = \{S_j \in \bar{\mathcal{S}} \mid t_j = c\}$
- suffix array is $\bar{\mathcal{S}}_a \mathcal{S}_a \bar{\mathcal{S}}_b \mathcal{S}_b \ldots$
- proof: $\bar{\mathcal{S}}_c \ni cab < ccc\ldots < cccd \in \mathcal{S}_c$

Merging in A2 and A3

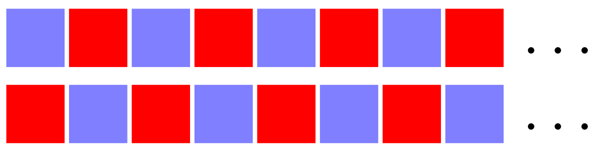
Problem: comparing sample and nonsample suffixes

= sample position = nonsample position

A2: Comparing odd and even suffixes

even

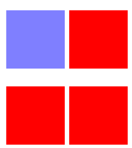
odd



A3: Comparing 0-suffixes and 1-suffixes

0-suffix

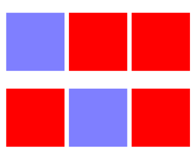
1-suffix



Comparing 0-suffixes and 2-suffixes

0-suffix

2-suffix



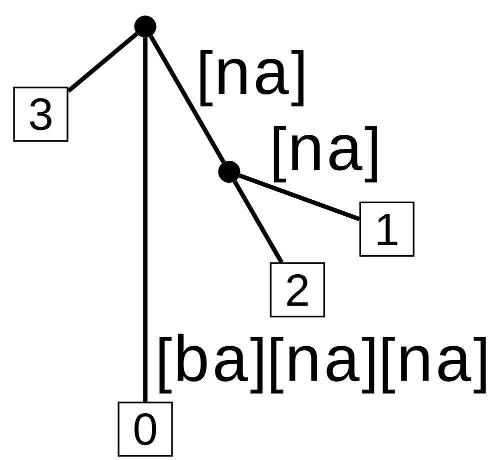
Step 1: Sort the sample

1. construct text R whose suffixes exactly represent sample $\mathcal{S}$
   - let $\mathcal{S} = \{S_{i_1}, S_{i_2}, S_{i_3}, \ldots\}$ with $i_1 < i_2 < i_3 < \cdots$
   - natural choice: $R = [t_{i_1} \ldots t_{i_2-1}][t_{i_2} \ldots t_{i_3-1}][t_{i_3} \ldots t_{i_4-1}] \ldots$
2. rename characters of R with ranks $\implies$ alphabet $[1, |R|]$
3. sort suffixes of R (recursion)

A2: $\mathcal{S} = \{S_i \mid i \bmod 2 \neq 0\}$

- $R = [t_1 t_2][t_3 t_4] \ldots$

$R =$ [ba][na][na]



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A2: $\mathcal{S} = \{S_i \mid i \bmod 2 \neq 0\}$
- $R = [t_1 t_2][t_3 t_4] \ldots$

A3: $\mathcal{S} = \{S_i \mid i \bmod 3 \neq 0\}$
- $R \neq [t_1][t_2 t_3][t_4][t_5 t_6] \ldots$

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   - let $\mathcal{S} = \{S_{i_1}, S_{i_2}, S_{i_3}, \ldots\}$ with $i_1 < i_2 < i_3 < \cdots$
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   - proper prefix problem: [a][a…] < [ab][…] < [a][c…]
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   - natural choice: $R = [t_{i_1} \ldots t_{i_2-1}][t_{i_2} \ldots t_{i_3-1}][t_{i_3} \ldots t_{i_4-1}] \ldots$
2. rename characters of R with ranks $\Longrightarrow$ alphabet $[1, |R|]$
   - proper prefix problem: $[\mathsf{a}][\mathsf{a} \ldots] < [\mathsf{ab}][\ldots] < [\mathsf{a}][\mathsf{c} \ldots]$
3. sort suffixes of R (recursion)

A2: $\mathcal{S} = \{S_i \mid i \bmod 2 \neq 0\}$

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A3: $\mathcal{S} = \{S_i \mid i \bmod 3 \neq 0\}$

- $R = [t_1 t_2 t_3][t_4 t_5 t_6] \ldots [t_2 t_3 t_4][t_5 t_6 t_7] \ldots$

Step 1: Sort the sample

1. construct text R whose suffixes exactly represent sample $\mathcal{S}$
   - let $\mathcal{S} = \{S_{i_1}, S_{i_2}, S_{i_3}, \ldots\}$ with $i_1 < i_2 < i_3 < \cdots$
   - natural choice: $R = [t_{i_1} \ldots t_{i_2-1}][t_{i_2} \ldots t_{i_3-1}][t_{i_3} \ldots t_{i_4-1}] \ldots$
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   - $R = [t_1 t_2 t_3][t_4 t_5 t_6] \ldots [t_2 t_3 t_4][t_5 t_6 t_7] \ldots$

Ax: $\mathcal{S} = \{S_i \mid S_i < S_{i+1}\}$
   - $R = [t_{i_1} \ldots t_{i_2-1} t_{i_2} \infty][t_{i_2} \ldots t_{i_3-1} t_{i_3} \infty][t_{i_3} \ldots t_{i_4-1} t_{i_4} \infty] \ldots$

Step 1: Sort the sample: Example

$$\begin{array}{l} \text{0 1 2 3 4 5} \\ S = \text{banana} \end{array}$$

A2: $\mathcal{S} = \{S_i \mid i \bmod 2 \neq 0\}$

1	anana	$R =$ [an][an][a]
3	ana	[an][a]
5	a	[a]

A3: $\mathcal{S} = \{S_i \mid i \bmod 3 \neq 0\}$

1	anana	$R =$ [ana][na][nan][a]
2	nana	[nan][a]
4	na	[na][nan][a]
5	a	[a]

Ax: $\mathcal{S} = \{S_i \mid S_i < S_{i+1}\}$

1	anana	$R =$ [ana∞][ana]
3	ana	[ana]

Step 2: Sort other suffixes $\bar{\mathcal{S}}$

- Let $next(\bar{\mathcal{S}}) = \{S_{j+1} \mid S_j \in \bar{\mathcal{S}}\}$ and $\bar{\mathcal{S}}_c = \{S_j \in \bar{\mathcal{S}} \mid t_j = c\}$
- For each $S_i \in next(\bar{\mathcal{S}})$ in sorted order
  insert S_{i-1} into $\bar{\mathcal{S}}_c$ with $c = t_{i-1}$

A2: $\mathcal{S} = \{S_i \mid i \bmod 2 \neq 0\}$ $\qquad$ $\bar{\mathcal{S}} = \{S_j \mid j \bmod 2 = 0\}$

- $next(\bar{\mathcal{S}}) = \mathcal{S}$

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A3: $\mathcal{S} = \{S_i \mid i \bmod 3 \neq 0\}$ $\quad\quad$ $\bar{\mathcal{S}} = \{S_j \mid j \bmod 3 = 0\}$

- $next(\bar{\mathcal{S}}) \subset \mathcal{S}$

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- $next(\bar{\mathcal{S}}) \subset \mathcal{S}$

Ax: $\mathcal{S} = \{S_i \mid S_i < S_{i+1}\}$ $\quad$ $\bar{\mathcal{S}} = \{S_j \mid S_j > S_{j+1}\}$

- scan suffix array $\epsilon \bar{\mathcal{S}}_a \mathcal{S}_a \bar{\mathcal{S}}_b \mathcal{S}_b \ldots$
- if suffix S_i is in $next(\bar{\mathcal{S}})$ insert S_{i-1}
- when scan reaches $S_j \in \bar{\mathcal{S}}$ it is already in place
  because $S_{j+1} < S_j$

Implementating A3: Subroutines

```
// compare pairs and triples
inline bool leq(int a1, int a2, int b1, int b2)
{ return(a1 < b1 || a1 == b1 && a2 <= b2); }
inline bool leq(int a1, int a2, int a3, int b1, int b2, int b3)
{ return(a1 < b1 || a1 == b1 && leq(a2,a3, b2,b3)); }

// radix sort (one pass)
static void radixPass(int* a, int* b, int* r, int n, int K)
{
   // count occurrences
   int* c = new int[K + 1];                       // counter array
   for (int i = 0; i <= K; i++) c[i] = 0;         // reset counters
   for (int i = 0; i < n; i++) c[r[a[i]]]++;      // count occurrences
   for (int i = 0, sum = 0; i <= K; i++)          // exclusive prefix sums
   { int t = c[i]; c[i] = sum; sum += t; }
   // sort
   for (int i = 0; i < n; i++) b[c[r[a[i]]]++] = a[i];
   delete [] c;
}
```

Implementating A3: Main function

```
// compute suffix array of s
// require s[n]=s[n+1]=s[n+2]=0, n>=2
void suffixArray(int* s, int* SA, int n, int K) {

   // initialize
   int n0=(n+2)/3, n1=(n+1)/3, n2=n/3, n02=n0+n2;
   int* s12 = new int[n02 + 3]; s12[n02]= s12[n02+1]=s12[n02+2]=0;
   int* SA12 = new int[n02 + 3]; SA12[n02]=SA12[n02+1]=SA12[n02+2]=0;
   int* s0 = new int[n0];
   int* SA0 = new int[n0];
```

Step 0: Compute sample
Step 1: Sort sample
Step 2: Sort other suffi xes
Step 3: Merge

```
   // clean up
   delete [] s12; delete [] SA12; delete [] SA0; delete [] s0;
}
```

Implementing A3: Step 0: Compute sample

```
// compute sample
for (int i=0, j=0; i < n+(n0-n1); i++) if (i%3 != 0) s12[j++] = i;
```

Implementing A3: Step 1: Sort the sample

```
// sort supercharacters (triples)
radixPass(s12 , SA12, s+2, n02, K);
radixPass(SA12, s12 , s+1, n02, K);
radixPass(s12 , SA12, s , n02, K);

// construct recursive text
int name = 0, c0 = -1, c1 = -1, c2 = -1;
for (int i = 0; i < n02; i++) {
   if (s[SA12[i]] != c0 || s[SA12[i]+1] != c1 || s[SA12[i]+2] != c2)
   { name++; c0 = s[SA12[i]]; c1 = s[SA12[i]+1]; c2 = s[SA12[i]+2];}
   if (SA12[i] % 3 == 1) { s12[SA12[i]/3] = name; }   // first half
   else { s12[SA12[i]/3 + n0] = name; }                // second half
}

if (name < n02) { // recurse if all supercharacters are not unique
   suffixArray(s12, SA12, n02, name);
   for (int i = 0; i < n02; i++) s12[SA12[i]] = i + 1;
} else   // end of recursion:  supercharacters are all unique
   for (int i = 0; i < n02; i++) SA12[s12[i] - 1] = i;
```

Implementing A3: Step 2: Sort other suffixes

```
// construct nonsample in order of next(nonsample)
for (int i=0, j=0; i < n02; i++)
   if (SA12[i] < n0) s0[j++] = 3*SA12[i];
// sort stably by first character
radixPass(s0, SA0, s, n0, K);
```

Implementing A3: Step 3: Merge

```
   // merge sample and nonsample suffixes
   for (int p=0, t=n0-n1, k=0; k < n; k++) {
#define GetI() (SA12[t] < n0 ?  SA12[t]*3+1 :  (SA12[t]-n0)*3+2)
      int i = GetI();
      int j = SA0[p];
      if (SA12[t] < n0 ?  // compare
         leq(s[i], s12[SA12[t] + n0], s[j], s12[j/3]) :
         leq(s[i],s[i+1],s12[SA12[t]-n0+1], s[j],s[j+1],s12[j/3+n0]))
      {          // sample suffix is smaller
         SA[k] = i; t++;
         if (t == n02)   // done --- only nonsample suffixes left
            for (k++; p < n0; p++, k++) SA[k] = SA0[p];
      } else {  // nonsample suffix is smaller
         SA[k] = j; p++;
         if (p == n0)    // done --- only sample suffixes left
            for (k++; t < n02; t++, k++) SA[k] = GetI();
      }
   }
```

Concluding remarks

- Implementation
  - **A3** and **Ax** are practical algorithms
  - can be made space-efficient
- Other models of computation
  - **A3** is easily parallelizable and externalizable
  - improved BSP and EREW-PRAM algorithms [K & Sanders, '03]
  - fast external memory implementation [Dementiev & al, '05]
- Related construction algorithms
  - $\mathcal{O}(vn + n\log n)$ time, $\mathcal{O}(n/\sqrt{v})$ extra space ($v \in [3, n]$)
    fast and space-efficient in practice [Burkhardt & K, '03]
  - $\mathcal{O}(vn)$ time, $\mathcal{O}(n/\sqrt{v})$ extra space [K & Sanders]

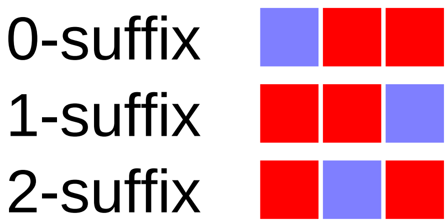
Open problems

- Suffix array has emerged from the shadow of suffix tree
  - several recent algoritms
  - missing algorithms?
- I still don't understand suffix arrays!
  - surprising algorithms
  - common combinatorial principles?
  - more surprises coming?

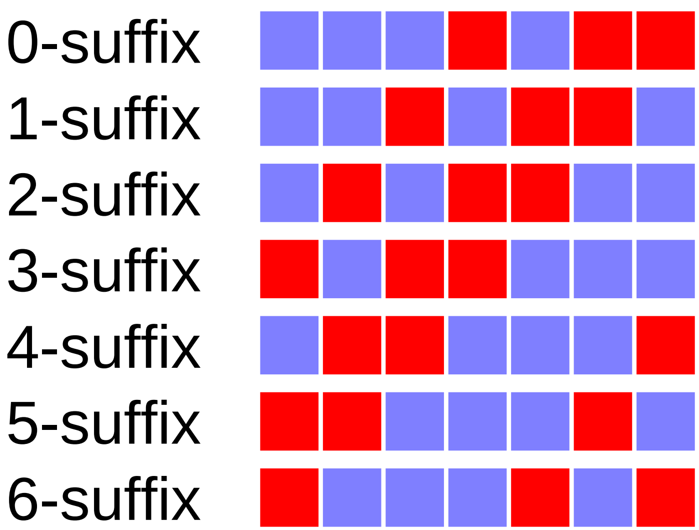
Difference cover samples

= sample position = nonsample position

A3: $\mathcal{S} = \{S_i \mid i \bmod 3 \in \{1,2\}\}$



A7: $\mathcal{S} = \{S_i \mid i \bmod 7 \in \{3,5,6\}\}$



Difference cover samples

$D \subseteq [0, v)$ is a difference cover modulo v if

$$\{i - j \bmod v \mid i, j \in D\} = [0, v)$$

- $D = \{1, 2\}$ is a difference cover modulo 3
- $D = \{3, 5, 6\}$ is a difference cover modulo 7
- $D = \{1\}$ is not a difference cover modulo 2

Algorithms

- **A3**
- $\mathcal{O}(vn + n \log n)$ time, $\mathcal{O}(n/\sqrt{v})$ extra space [Burkhardt & K, '03]
- $\mathcal{O}(vn)$ time, $\mathcal{O}(n/\sqrt{v})$ extra space [K & Sanders, ??]