Diagrammatic sets and homotopically sound rewriting

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Algebraic Rewriting Seminar
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- *Diagrammatic sets and rewriting in weak higher categories*, arXiv:2007.14505
- *The smash product of monoidal theories*, arXiv:2101.10361

A key insight of polygraph theory:

Rewriting theory
as a theory of
directed cell complexes

(a kind of combinatorial topology of *directed spaces*)

Models of cell complexes

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1. models of n-cells (and their $(n-1)$-boundaries);
2. models of "gluing maps" specifying how n-cells are put together

Models of cell complexes

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- **synthetic** models, as *higher inductive types* – cells are constructors of identity types, gluing is specified by the type theory

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Directed type theories may give us synthetic models, but are at a quite primordial stage...

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- Polygraphs are very expressive!
- None of the other models are very expressive, rewriting-wise. Point-set models can do direction only on 1-cells. Typical combinatorial models are limiting in terms of the shape of generators.

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- For point-set models, it's obvious. Combinatorial models usually have nice geometric realisations that satisfy this.
- Polygraphs do **not** satisfy this: not all gluing maps (modelled by arbitrary functors of ω-categories) have a sound topological interpretation.

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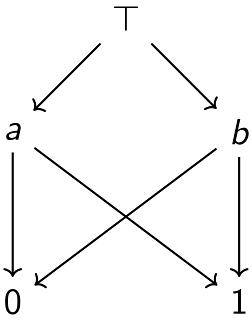
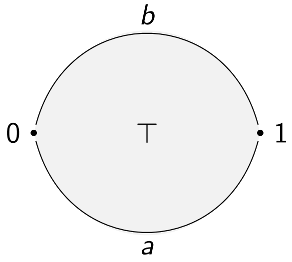
- a notorious Kapranov–Voevodsky 1991 paper (name is due to them)

Face posets

We can associate to a cell complex its face poset...





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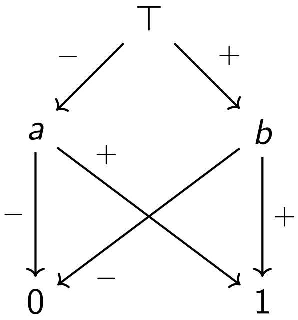

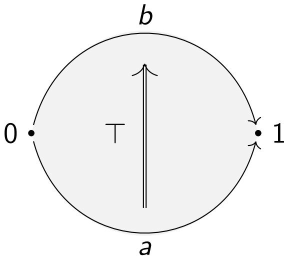

and to a higher-categorical pasting diagram its oriented face poset.

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Regular CW complexes are essentially combinatorial objects.

The face poset of a regular CW n-ball is
a combinatorial model of an n-cell.

Face posets

The diagrammatic set model of directed cell complex:

- Directed n-cells are modelled by regular directed complexes

  (oriented face posets of pasting diagrams, whose underlying poset is the face poset of a regular CW complex)

  with a greatest element of rank n

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- Gluing is given by maps of posets that are compatible functorially with both realisations

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- An *orientation* on a finite poset P is an edge-labelling $o : \mathscr{H} P_1 \to \{+, -\}$ of its Hasse diagram.
- An *oriented graded poset* is a finite graded poset with an orientation.
- If $U \subseteq P$ is (downward) closed, $\alpha \in \{+, -\}$, $n \in \mathbb{N}$,

$$\Delta_n^\alpha U := \{x \in U \,|\, \dim(x) = n \text{ and if } y \in U \text{ covers } x\text{, then } o(y \to x) = \alpha\},$$
$$\partial_n^\alpha U := \mathrm{cl}(\Delta_n^\alpha U) \cup \{x \in U \,|\, \text{for all } y \in U\text{, if } x \le y\text{, then } \dim(y) \le n\},$$
$$\Delta_n U := \Delta_n^+ U \cup \Delta_n^- U, \qquad \partial_n U := \partial_n^+ U \cup \partial_n^- U.$$

Directed complexes

If U is a closed subset of P, then U is a *molecule* if either

- U has a greatest element, in which case we call it an *atom*, or
- there exist molecules U_1 and U_2, both properly contained in U, and $n \in \mathbb{N}$ such that $U_1 \cap U_2 = \partial_n^+ U_1 = \partial_n^- U_2$ and $U = U_1 \cup U_2$.

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An oriented graded poset P is a *directed complex* if, for all $x \in P$ and $\alpha, \beta \in \{+, -\}$, if $n = \mathrm{dim}(x)$,

1. $\partial^\alpha x$ is a molecule, and
2. $\partial^\alpha(\partial^\beta x) = \partial^\alpha_{n-2} x$.

Regular directed complexes

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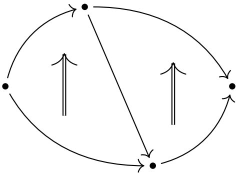
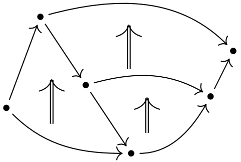
*simplicial nerve of poset + realisation of simplicial sets

Spherical boundary





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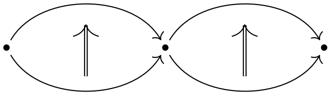

$\mathcal{C}$-directed complexes

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- A *$\mathcal{C}$-directed complex* is a directed complex whose atoms are all in $\mathcal{C}$.

Morphisms of directed complexes

A *map* $f : P \to Q$ of $\mathcal{C}$-directed complexes is a function that satisfies

$$\partial_n^\alpha f(x) = f(\partial_n^\alpha x)$$

for all $x \in P$, $n \in \mathbb{N}$, and $\alpha \in \{+, -\}$.

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Let $f : P \to Q$ be a map. Then f is a closed, order-preserving, dimension-non-increasing function of the underlying posets.

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A $\mathcal{C}$-functor factors e.u. as a *subdivision* followed by an *inclusion*.

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A span of inclusions of subcategories:

$$\mathbf{DCpx}^{\mathcal{C}} \hookleftarrow \mathbf{DCpx}^{\mathcal{C}}_{in} \hookrightarrow \mathbf{DCpx}^{\mathcal{C}}_{fun}$$

Convenient classes

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The class $\mathcal{S}$ is convenient!

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- A *diagram of shape* U *in* X is a morphism $x : U \to X$ where U is a molecule.
- It is *composable* if $U \in \mathcal{C}$, and a *cell* if U is an atom.

Diagrammatic sets

A *diagrammatic complex* is a diagrammatic set X together with a set $\mathscr{X} = \sum_{n \in \mathbb{N}} \mathscr{X}_n$ of *generating* cells such that, for all $n \in \mathbb{N}$,

$$\begin{array}{ccc} \coprod_{x \in \mathscr{X}_n} \partial U(x) & \hookrightarrow & \coprod_{x \in \mathscr{X}_n} U(x) \\ \downarrow & & \downarrow (x)_{x \in \mathscr{X}_n} \\ \sigma_{\leq n-1} X & \hookrightarrow & \sigma_{\leq n} X \end{array}$$

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This is our model of a directed cell complex.

Homotopical soundness (and completeness)

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- The realisation of a diagrammatic complex $(X, \mathscr{X})$ is a CW complex with one generating cell for each cell in $\mathscr{X}$.

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- Moreover, the sequence of homotopy groups of a space X can be read from a combinatorially defined sequence of homotopy groups of SX.

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The price we pay for homotopical soundness is that "empty space" (sometimes) has to be explicitly handled.

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$\odot$**Set** is equivalent to the category $\mathrm{PSh}_{\Gamma}(\mathbf{DCpx}^{\mathcal{C}})$ of Γ-continuous presheaves on $\mathbf{DCpx}^{\mathcal{C}}$.

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Conjecture

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  $\rightsquigarrow$ Computational analyses should be relative to sub-presheaves of the underlying combinatorial polygraph.
- Taking the "free non-unital ω-category" is a way of capturing the transitive closure of the rewrite relation

Equivalences and weak composites

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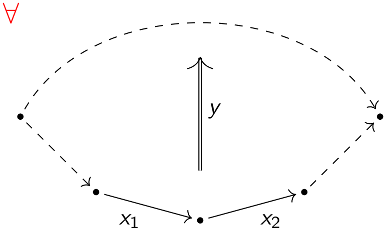

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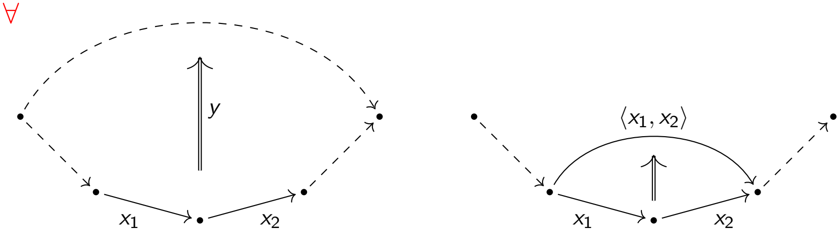

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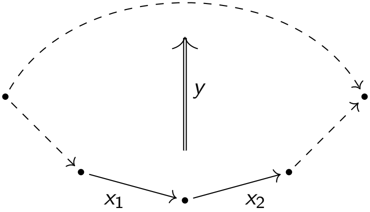

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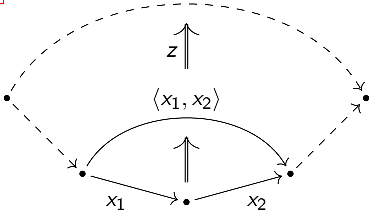

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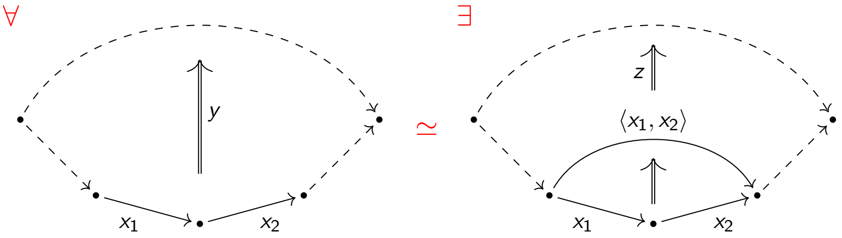

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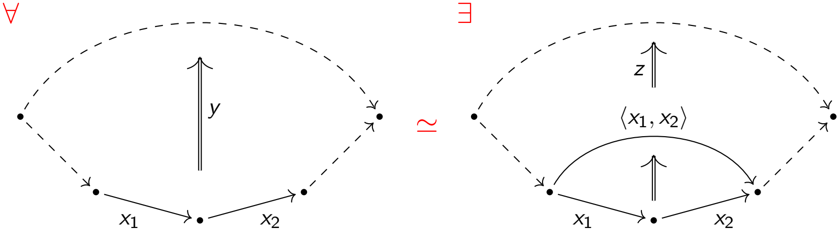

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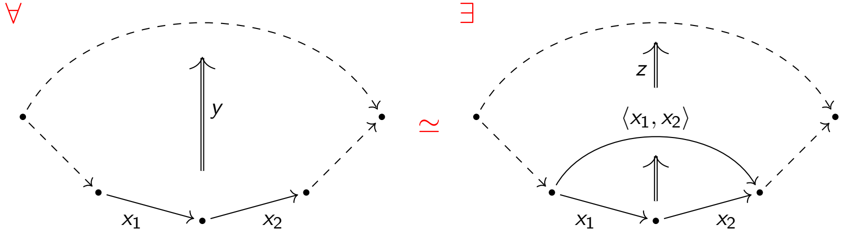

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whose definition involves 4-dimensional equivalence diagrams, etc

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- Morphisms of diagrammatic sets preserve equivalences.
- If X is a space, every diagram in SX is an equivalence.

Now, if there is time, an application
(from the more recent paper)

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This is part of a monoidal structure on $\mathbf{DCpx}^{\mathcal{C}}$,
which restricts to $\odot$,
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This allows us to define a (Gray) smash product $(X, \bullet_X) \oslash (Y, \bullet_Y)$ of pointed diagrammatic sets,
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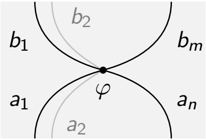

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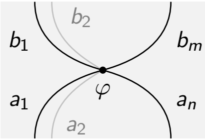

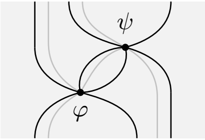

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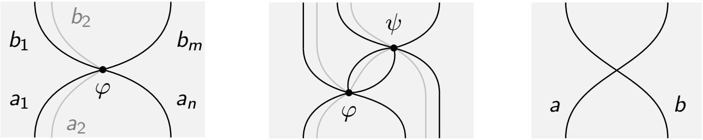

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Models of $(T, \mathscr{T})$ in $\mathbf{M}$ form a category $\mathrm{Mod}_{\mathbf{M}}(T, \mathscr{T})$ with monoidal natural transformations as morphisms.

This category admits a symmetric monoidal structure.

(Idea: "run operations in parallel", use symmetry to redistribute inputs and outputs as needed)

The tensor product of props

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The tensor product $(T, \mathscr{T}) \otimes_{\mathbb{S}} (S, \mathscr{S})$ is determined universally by the requirement that

- models of $(T, \mathscr{T}) \otimes_{\mathbb{S}} (S, \mathscr{S})$ in $\mathbf{M}$

correspond naturally to

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There is

- an embedding **Prop** $\hookrightarrow$ **Prob**, and
- a forgetful functor $\mathrm{U}\colon$ **Prob** $\to$ **Pro**,

with left adjoints $\mathrm{r}\colon$ **Prob** $\to$ **Prop** and $\mathrm{F}\colon$ **Pro** $\to$ **Prob**.

The external product of pros

There is an *external* tensor product $- \otimes - : \mathbf{Pro} \times \mathbf{Pro} \to \mathbf{Prob}$

$$\begin{array}{ccc} \mathbf{Pro} \times \mathbf{Pro} & \xrightarrow{\otimes} & \mathbf{Prob} \\ {\scriptstyle \mathsf{rF} \times \mathsf{rF}} \downarrow & & \downarrow {\scriptstyle \mathsf{r}} \\ \mathbf{Prop} \times \mathbf{Prop} & \xrightarrow[\otimes_{\mathbb{S}}]{} & \mathbf{Prop} \end{array}$$

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We recover the tensor product of props from the external product of their underlying pros, by imposing that a natural family of inclusions of the factors into their product preserve braidings.

Diagrammatic sets, pros, and probs

There are adjunctions relating

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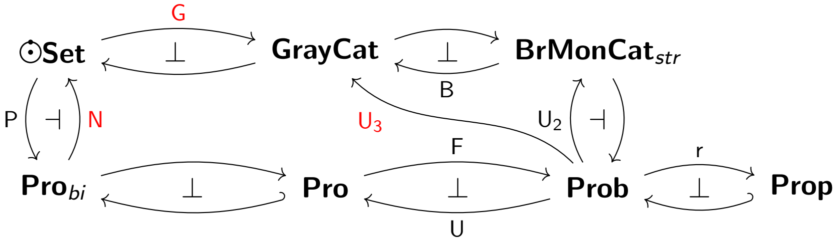

The external product of pros is a smash product

Theorem

The diagram of functors

$$
\begin{array}{ccccc}
\mathbf{Pro} \times \mathbf{Pro} & \xrightarrow{\;- \otimes -\;} & \mathbf{Prob} & & \\
 & & & \searrow^{U_3} & \\
\Big\downarrow {\scriptstyle N \times N} & & & & \mathbf{GrayCat} \\
 & & & \nearrow_{G} & \\
\odot\mathbf{Set}_\bullet \times \odot\mathbf{Set}_\bullet & \xrightarrow[\;- \owedge (-)^\circ\;]{} & \odot\mathbf{Set}_\bullet & &
\end{array}
$$

commutes up to natural isomorphism.

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- If X and Y have interesting oriented n-cells, then $X \owedge Y$ has interesting oriented k-cells up to $k = 2n$!

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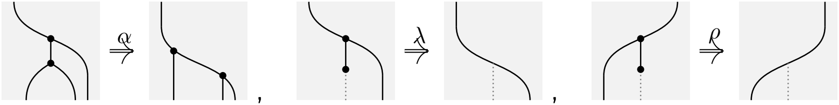
Idea: Given presentations X of $(T, \mathscr{T})$ and Y of $(S, \mathscr{S})$, the smash product $X \owedge Y^\circ$ produces

1. a presentation (with oriented equations) of $(T, \mathscr{T}) \otimes (S, \mathscr{S})$,
2. **plus** higher-dimensional coherence cells, or oriented *syzygies*, for this presentation.

Towards compositional higher rewriting

Let X be a presentation of the theory of monoids *Mon* with the 3-cells



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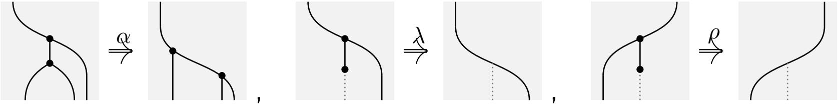
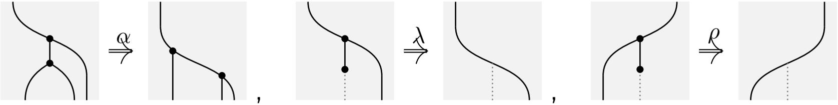


Then $X \owedge X$ is a presentation of $\mathit{Mon} \otimes \mathit{Mon}^{\mathrm{co}}$, the theory of *bialgebras*.

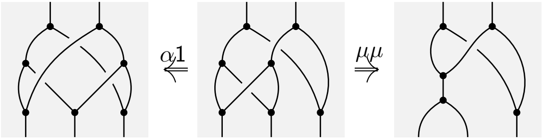
Towards compositional higher rewriting

Let X be a presentation of the theory of monoids *Mon* with the 3-cells



Then $X \owedge X$ is a presentation of $\mathit{Mon} \otimes \mathit{Mon}^{\mathrm{co}}$, the theory of *bialgebras*.

It has the following “new” critical branching:

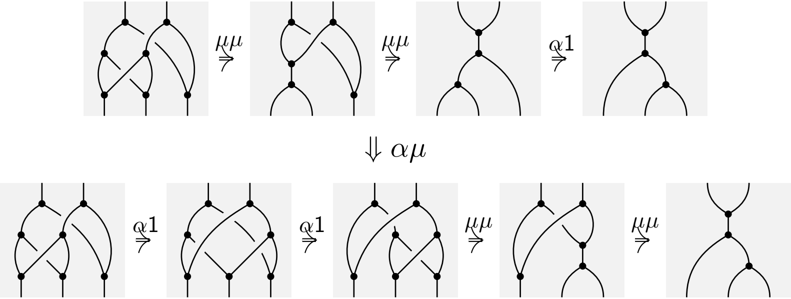


Towards compositional higher rewriting

The 5-cell $\alpha \otimes \mu$ in $X \owedge X$ exhibits confluence at this critical branching:

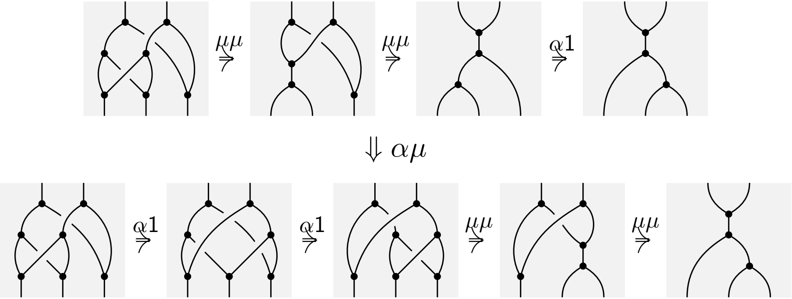
Towards compositional higher rewriting

The 5-cell $\alpha \otimes \mu$ in $X \oslash X$ exhibits confluence at this critical branching:



Towards compositional higher rewriting

The 5-cell $\alpha \otimes \mu$ in $X \owedge X$ exhibits confluence at this critical branching:



6-cells such as $\alpha \otimes \alpha$ are *higher syzygies* exhibiting confluence at critical branchings of syzygies

Towards compositional higher rewriting

Question:

If we start from presentations with nice computational properties or nice homotopical properties,

do we obtain nice presentations of their tensor product?