ZW calculi: Diagrammatic languages for quantum computing

Amar Hadzihasanovic

RIMS, Kyoto University

Dubrovnik, 24 September 2018

References

LICS 2018

Two complete axiomatisations of pure-state qubit quantum computing, with K. F. Ng and Q. Wang

FSCD 2018

A diagrammatic axiomatisation of fermionic quantum circuits, with G. de Felice and K. F. Ng

What this is about, in short

- Equational axiomatisations of the theory of *extensional equality* of certain quantum circuits
- Which also provide, *topologically*, information on the correlations between different parts of a circuit

What this is about, in short

- Equational axiomatisations of the theory of *extensional equality* of certain quantum circuits
- Which also provide, *topologically*, information on the correlations between different parts of a circuit

The interface between these two aspects is given by monoidal categories presented by string diagrams

From algebraic theories to PROs to string diagrams

Presentation of an **algebraic theory**: finitary operations + identities (with all variables universally quantified)

Example: theory of abelian groups

Binary multiplication $m(-,-)$, unary inverse $i(-)$, nullary unit u

$$m(m(x,y),z) = m(x,m(y,z)), \qquad m(x,u) = x = m(u,x),$$

$$m(x,i(x)) = u = m(i(x),x), \qquad m(x,y) = m(y,x)$$

From algebraic theories to PROs to string diagrams

Lawvere '63: the same information presents particular *categories with finite products*

Example: Lawvere theory of abelian groups

Generating morphisms $m : a \times a \to a$, $i : a \to a$, $u : 1 \to a$

From algebraic theories to PROs to string diagrams

Lawvere '63: the same information presents particular *categories with finite products*

Example: Lawvere theory of abelian groups

Generating morphisms $m : a \times a \to a$, $i : a \to a$, $u : 1 \to a$
+ structural morphisms $s : a \times a \to a \times a$, $c : a \to a \times a$, $d : a \to 1$

$(m \times \mathrm{id}_a); m = (\mathrm{id}_a \times m); m,$ $\qquad$ $(\mathrm{id}_a \times u); m = \mathrm{id}_a = (u \times \mathrm{id}_a); m$

$c; (\mathrm{id}_a \times i); m = d; u = c; (i \times \mathrm{id}_a); m,$ $\qquad$ $m = s; m$

From algebraic theories to PROs to string diagrams

Lawvere '63: the same information presents particular *categories with finite products*

Example: Lawvere theory of abelian groups

Generating morphisms $m : a \times a \to a$, $i : a \to a$, $u : 1 \to a$
+ structural morphisms $s : a \times a \to a \times a$, $c : a \to a \times a$, $d : a \to 1$

$$(m \times \mathrm{id}_a); m = (\mathrm{id}_a \times m); m, \qquad (\mathrm{id}_a \times u); m = \mathrm{id}_a = (u \times \mathrm{id}_a); m$$

$$c; (\mathrm{id}_a \times i); m = d; u = c; (i \times \mathrm{id}_a); m, \qquad m = s; m$$

plus whatever is needed to make $\times$ a categorical product...

From algebraic theories to PROs to string diagrams

Why treat the "structural" morphisms differently?

From algebraic theories to PROs to string diagrams

Why treat the "structural" morphisms differently?

Example: PRO of commutative Hopf algebras

Generating morphisms $m : a \otimes a \to a$, $i : a \to a$, $u : 1 \to a$, $s : a \otimes a \to a \otimes a$, $c : a \to a \otimes a$, $d : a \to 1$

$$(m \otimes \mathrm{id}_a); m = (\mathrm{id}_a \otimes m); m, \qquad (\mathrm{id}_a \otimes u); m = \mathrm{id}_a = (u \otimes \mathrm{id}_a); m$$

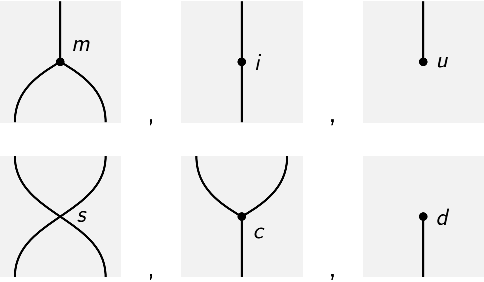
$$c = c; s \qquad c; (\mathrm{id}_a \otimes i); m = d; u = c; (i \otimes \mathrm{id}_a); m, \qquad m = s; m$$

$$c; (m \otimes \mathrm{id}_a) = c; (\mathrm{id}_a \otimes c), \qquad c; (\mathrm{id}_a \otimes d) = \mathrm{id}_a = c; (d \otimes \mathrm{id}_a)$$

plus other equations ensuring s behaves like a swap, m and c interact as expected, etc

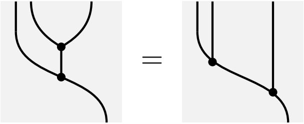
From algebraic theories to PROs to string diagrams

Formally, Joyal-Street '91 (informally, way before?)

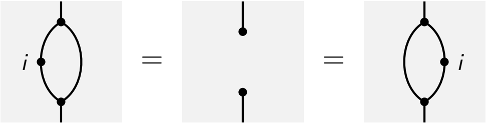


From algebraic theories to PROs to string diagrams

Coassociativity:

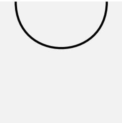
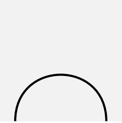


Inverse:

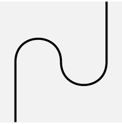
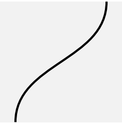
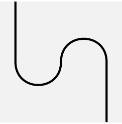
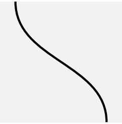


Self-dual objects and "undirectedness"

At the other end of the spectrum w.r.t. "cartesian"...

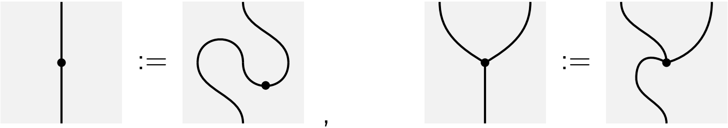
 , 

satisfying

 =  ,  = 

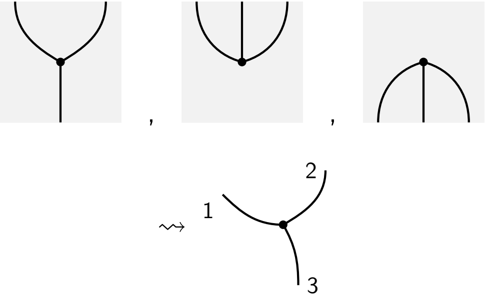
Self-dual objects and "undirectedness"

With self-duality, we can turn inputs into outputs



If permuting the inputs and outputs of a generator, the result *only depends on the arity* of the resulting diagram, then we can effectively treat the diagram as an undirected graph

Self-dual objects and “undirectedness”



Theory can be studied with methods of graph rewriting

The pure-state qubit model

Single system: two-dimensional Hilbert space (with a fixed "computational" basis $|\,0\,\rangle$, $|\,1\,\rangle$)

Composite system: tensor product of Hilbert spaces

The pure-state qubit model

Single system: two-dimensional Hilbert space (with a fixed "computational" basis $|0\rangle, |1\rangle$)
Composite system: tensor product of Hilbert spaces

The only physical processes are **isometries**!

However restricting to isometries may not give the best portrayal of what's "logically" happening, so we consider **all linear maps**

The monoidal category Qubit

Morphisms $n \to m$ are linear maps $(\mathbb{C}^2)^{\otimes n} \to (\mathbb{C}^2)^{\otimes m}$

Brief chronology of string diagrams for **Qubit**

- 2004 Abramsky, Coecke — categorical quantum mechanics
- 2008 Coecke, Duncan — first "ZX calculus" axioms

ZX calculus: two colours of vertices, axioms symmetric in the two; decomposition of CNOT gate

- 2014 Backens — complete axioms for *stabiliser fragment*

Brief chronology of string diagrams for **Qubit**

- 2004 Abramsky, Coecke — categorical quantum mechanics
- 2008 Coecke, Duncan — first "ZX calculus" axioms

ZX calculus: two colours of vertices, axioms symmetric in the two; decomposition of CNOT gate

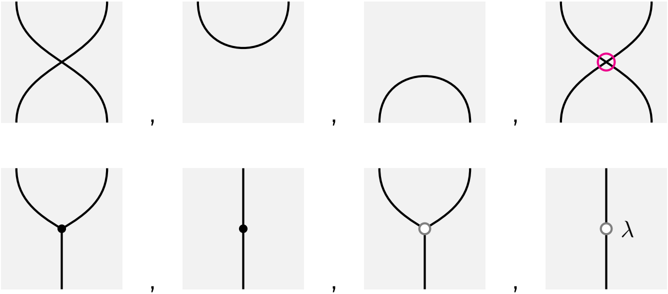
- 2014 Backens — complete axioms for *stabiliser fragment*

Meanwhile:

- 2010 Coecke, Kissinger — propose an alternative presentation, with two colours related to "inequivalent" (in a specific, operational sense) three-qubit states

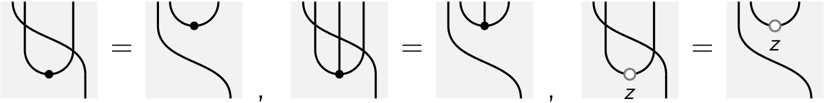
Qubit ZW calculus

The **qubit ZW calculus** is a presentation of **Qubit** with generators



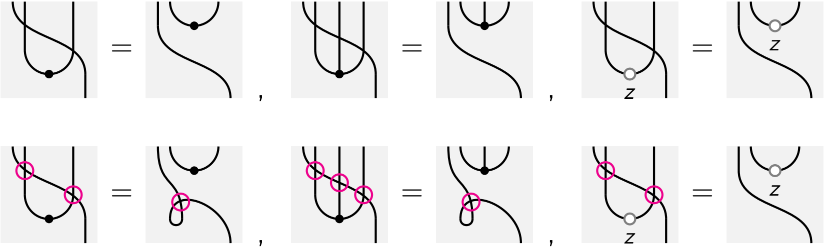
Qubit ZW calculus

The two "swaps":



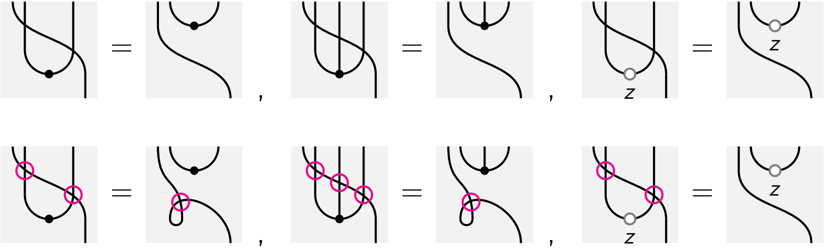
Qubit ZW calculus

The two “swaps”:

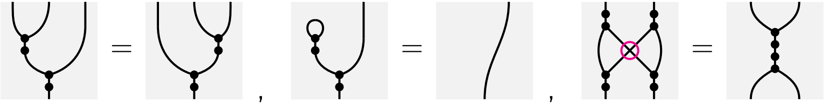


Qubit ZW calculus

The two “swaps”:



The black vertices:



From qubit to fermionic ZW

Unlike the ZX calculus,

- somewhat disconnected from "typical" qubit gates;
- no symmetry between basis states...

However,

From qubit to fermionic ZW

Unlike the ZX calculus,

- somewhat disconnected from "typical" qubit gates;
- no symmetry between basis states...

However,

We can interpret $|0\rangle$ and $|1\rangle$ as the *empty* and *occupied* states of a different physical system: a **local fermionic mode**, on which the Bravyi-Kitaev model of *fermionic quantum computation* is based

The pure-state fermionic model

Allowed operations are the ones that

- preserve the parity (number of particles mod 2), or
- introduce any number of particles into the system

The monoidal category LFM

Morphisms $n \to m$ are purely **even** or purely **odd** linear maps $(\mathbb{C}^2)^{\otimes n} \to (\mathbb{C}^2)^{\otimes m}$

The pure-state fermionic model

Allowed operations are the ones that

- preserve the parity (number of particles mod 2), or
- introduce any number of particles into the system

The monoidal category LFM

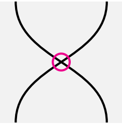
Morphisms $n \to m$ are purely **even** or purely **odd** linear maps $(\mathbb{C}^2)^{\otimes n} \to (\mathbb{C}^2)^{\otimes m}$

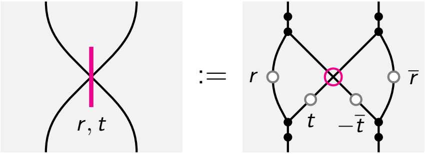
Example: $|\,0\,\rangle \mapsto |\,1\,\rangle$, or $|\,0\,\rangle \mapsto |\,00\,\rangle + |\,11\,\rangle$ is allowed;
$|\,0\,\rangle \mapsto |\,0\,\rangle + |\,1\,\rangle$ is not.

Fermionic ZW calculus

All generators of qubit ZW calculus except the ternary white vertex

"Natural" fermionic gates in the calculus:

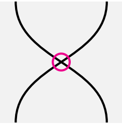


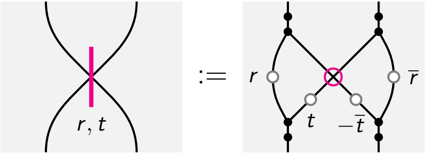


Fermionic ZW calculus

All generators of qubit ZW calculus except the ternary white vertex

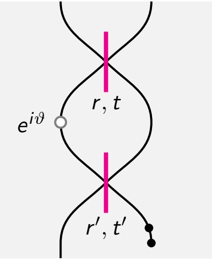
"Natural" fermionic gates in the calculus:



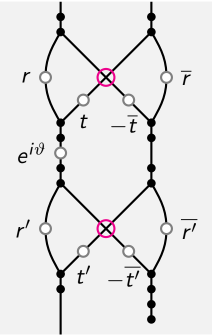


Also suggests generalisations to higher-dimensional systems

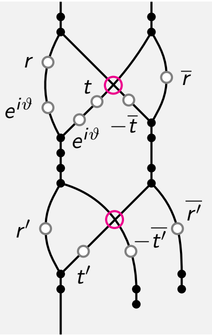
Mach-Zehnder interferometer



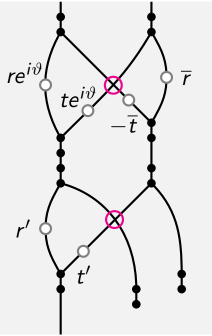
Mach-Zehnder interferometer



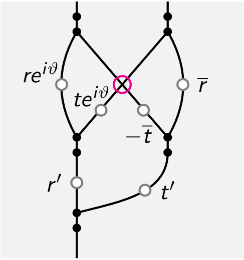
Mach-Zehnder interferometer



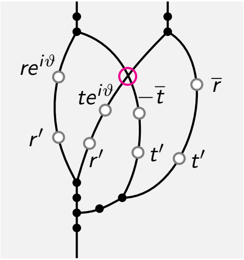
Mach-Zehnder interferometer



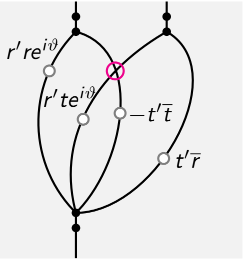
Mach-Zehnder interferometer



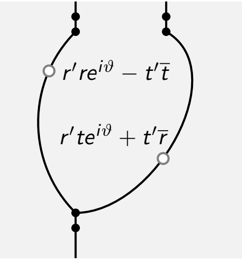
Mach-Zehnder interferometer



Mach-Zehnder interferometer



Mach-Zehnder interferometer



Qubit = fermionic + duplication?

The fermionic model is “resource sensitive”: there is no map duplicating particles.

Qubit = fermionic + duplication?

The fermionic model is "resource sensitive": there is no map duplicating particles.

However, it soundly embeds into the qubit model; to cover all qubit maps, it suffices to add a single generator, which can be interpreted as "copying particles" ($|\,1\,\rangle \mapsto |\,11\,\rangle$)

Qubit = fermionic + duplication?

The fermionic model is “resource sensitive”: there is no map duplicating particles.

However, it soundly embeds into the qubit model; to cover all qubit maps, it suffices to add a single generator, which can be interpreted as “copying particles” ($|1\rangle \mapsto |11\rangle$)

$$\text{fermionic} : \text{qubit} \sim \text{linear} : \text{intuitionistic?}$$