

in order to determine the angle between plates at equilibrium ($\theta_0 = 2\alpha_0$) we need to derive formula of potential energy as a function of α_0 .

It's obvious that the potential energy reaches minimum value at equilibrium.

$$h = \sqrt{\frac{m}{\rho L}} (\sin\alpha \cos\alpha)^{-\frac{1}{2}}, \quad \sqrt{\frac{m}{\rho L}} \approx 2\sqrt{\sin\alpha \cos\alpha}$$

Vertical position of CM relative to the ceiling: $y_{cm} = 2l \cos\alpha - \frac{2}{3} \sqrt{\frac{m}{\rho L}} (\tan\alpha)^{-\frac{1}{2}}$

Potential energy as function $\alpha \Rightarrow U(\alpha) = -mg \left(2l \cos\alpha - \frac{2}{3} \sqrt{\frac{m}{\rho L}} (\tan\alpha)^{-\frac{1}{2}} \right)$

Then, $\left. \frac{dU(\alpha)}{d\alpha} \right|_{\alpha=\alpha_0} = 0$

$$\sin^2 \alpha_0 = \frac{1}{6}$$

$$\alpha_0 = \sin^{-1} \frac{1}{\sqrt{6}}$$

$$\theta_0 = 2\alpha_0 = 48.19^\circ //$$

Now, we work in O's reference. Let \vec{u} denotes vector velocity of water in O's reference. Consider the motion of mass m.

$$m = \rho y^2 \tan\alpha$$

because our system is isolated, $\frac{\delta m}{\delta t}$ equal to zero.

$$0 = \frac{\delta}{\delta t} (\rho y^2 \tan\alpha)$$

$$\frac{dy}{dt} = - \frac{y \cdot \dot{\alpha}}{\sin^2 \alpha}$$

$$u_y = - \frac{y \cdot \dot{\alpha}}{\sin^2 \alpha}$$

Consider the motion of mass m.

$$m = 2\rho x L \left(l \cos\alpha - \frac{x}{\tan\alpha} \right)$$

$$\frac{dm}{dt} = 0$$

$$0 = \dot{x} \left(l \cos\alpha - \frac{x}{\tan\alpha} \right) + x \left(\frac{\delta}{\delta t} \left(l \cos\alpha - \frac{x}{\tan\alpha} \right) \right)$$

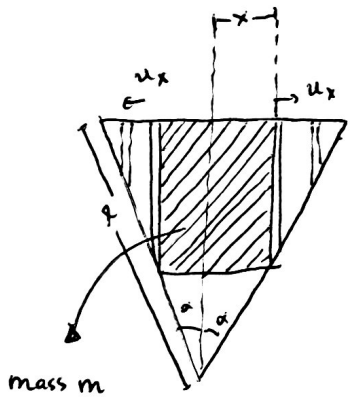
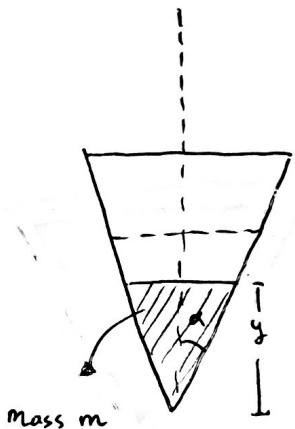
$$\dot{x} = \frac{\left(\frac{l}{2\sin\alpha} - \frac{x}{\sin^2\alpha} \right) x \dot{\alpha}}{\left(l \cos\alpha - \frac{2x}{\tan\alpha} \right)} = \frac{x}{2\sin\alpha \cos\alpha}$$

$$u_x = \dot{x} = \frac{x \cdot \dot{\alpha}}{\sin^2 \alpha}$$

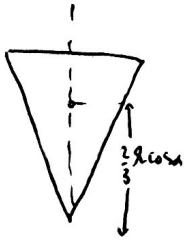
Alternative way: $\nabla \cdot \vec{u} = 0$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

$$\int_0^x \frac{\partial u_x}{\partial x} = \frac{1}{\sin^2 \alpha} \dot{\alpha} \int_0^x \partial x \Rightarrow u_x = \frac{x}{\sin^2 \alpha} \dot{\alpha}$$

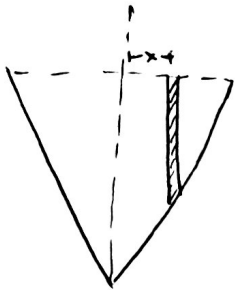


$$y_{cm} = \frac{2}{3} \ell \cos \alpha$$



Velocity in CM's frame: $\vec{v}_{rel} = \frac{x}{\sin 2\alpha} \hat{x} + \frac{y - \frac{2}{3} \ell \cos \alpha}{\sin 2\alpha} \hat{y}$

* Kinetic energy related to the horizontal component of \vec{v}_{rel}

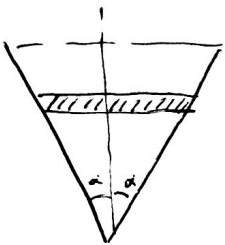


$$dE_{krel,x} = \frac{1}{2} \rho (\ell \cos \alpha - \frac{x}{\tan \alpha}) \frac{L x^2}{\sin^2 2\alpha} d^2 x$$

$$E_{krel,x} = \frac{2(\frac{1}{2}) \rho L}{\sin^2 2\alpha} \int_0^{\ell \sin \alpha} (\ell \cos \alpha - \frac{x}{\tan \alpha}) x^2 dx$$

$$= \frac{\rho L \ell^4 \alpha^2}{48} \tan \alpha$$

* Kinetic energy related to vertical component of \vec{v}_{rel}



$$dE_{krel,y} = \frac{1}{2} \rho (2y \tan \alpha) (dy) \frac{(y - \frac{2}{3} \ell \cos \alpha)^2}{\sin^2 2\alpha} L \alpha^2$$

$$E_{krel,y} = \frac{\rho L \tan \alpha}{\sin^2 2\alpha} \int_0^{\ell \cos \alpha} (y - \frac{2}{3} \ell \cos \alpha)^2 y dy$$

$$= \frac{\rho L \ell^4 \alpha^2}{144 \tan \alpha}$$

In lab frame, $E_{total} = E_{krel,x} + E_{krel,y} - mg \left(2\ell \cos \alpha - \frac{2}{3} \sqrt{\frac{m}{\rho L}} (\tan \alpha)^{-\frac{1}{2}} \right) + E_{k,pm}$

$E_{k,pm} \approx 0$
 so, $E_{total} = \frac{3 \tan^2 \alpha + 1}{144} \rho \ell^4 L \alpha^2 - mg \left(2\ell \cos \alpha - \frac{2}{3} \sqrt{\frac{m}{\rho L}} (\tan \alpha)^{-\frac{1}{2}} \right)$

Small oscillations $\rightarrow \alpha = \alpha_0 + \Delta \alpha$, $\Delta \alpha / \alpha_0 \ll 1$, $\alpha_0^3 \approx 0$, $\sin \Delta \alpha \approx \Delta \alpha$, and $\cos \Delta \alpha \approx 1$

$$\frac{dE_{total}}{dt} = 0$$

$$0 \approx \frac{3 \tan^2 \alpha_0 + 1}{72 \tan \alpha_0} \rho \ell^4 L \alpha_0 \ddot{\alpha} + mg \left(2\ell \cos \alpha_0 + \frac{\frac{3}{2} - 2 \sin^2 \alpha_0}{3 \sin^2 \alpha_0 \cos \alpha_0} \right) \alpha_0 \Delta \alpha$$

Substitute $m \approx \rho \ell \cdot L \sin \alpha_0 \cos \alpha_0$

$$0 = \ddot{\alpha} + \frac{72 \sin^2 \alpha_0}{3 \tan^2 \alpha_0 + 1} \left[2 \cos \alpha_0 + \frac{\frac{3}{2} - 2 \sin^2 \alpha_0}{3 \sin^2 \alpha_0 \cos \alpha_0} \right] \frac{g}{\ell} \Delta \alpha$$

$$\omega = \sqrt{\frac{72 \sin^2 \alpha_0}{3 \tan^2 \alpha_0 + 1} \left[2 \cos \alpha_0 + \frac{\frac{3}{2} - 2 \sin^2 \alpha_0}{3 \sin^2 \alpha_0 \cos \alpha_0} \right] \frac{g}{\ell}}$$

$$\omega = 5.733 \sqrt{\frac{g}{\ell}} //$$