Physics Cup Problem 4

Navneel Singhal

Abstract

Here I present the solution to the Problem 4 of the Physics Cup 2018.

Problem

A V-shaped vessel is made from two plates of width l and length $L \gg l$ which are connected via a frictionless hinge at its bottom. The vessel is fixed to a ceiling using light ropes of length l as shown in figure. The vessel is filled almost up to the rim with water of density ρ, and is subject to the homogeneous gravity field g. The mass of the ropes and plates is negligible. Find the angle between the plates, and the circular frequency of the lowest-frequency mirrorsymmetric oscillation mode of this system (evaluate the numerical prefactor of your expression with the precision of four significant digits). Neglect any water motion perpendicular to the plane of the figure.

Solution

Part (a) – The angle between the ropes

Let the angle between the ropes be $2\theta_0$. Due to the four equal lengths l, the cross-section along the motion is a rhombus.
We use the torque equation for the axis along the hinge for one of the plates, and the force equation for the system consisting of the plates and the water.

Suppose the tension in the ropes is T; it is equal in both ropes due to symmetry. The torque on the plate on the right due to tension equals $Tl\sin(2\theta_0)$.
Now consider the torque due to excess pressure in the fluid. Take an element of the plate at a distance r from the hinge and of width $\mathrm{d}r$. The force acting on it equals $\rho g(l-r)\cos\theta_0 L\mathrm{d}r$, and so the torque on the element is $r\rho g(l-r)\cos\theta_0 L\mathrm{d}r$. The magnitude of this torque is $\int_0^l r\rho g(l-r)\cos\theta_0 L\mathrm{d}r = \frac{1}{6}\rho g L l^2\cos\theta_0$.

The torque balance equation gives us that

$$Tl\sin(2\theta_0) = \frac{1}{6}\rho g L l^2 \cos\theta_0$$

The force equation gives us that

$$2T\cos\theta_0 = \frac{1}{2}\rho l^2 \sin(2\theta_0) L g$$

Eliminating T from these equations, we have $\sin^2\theta_0 = \frac{1}{6}$, so $\cos(2\theta_0) = \frac{2}{3}$. So the angle between the two ropes is $\boxed{\arccos\frac{2}{3}}$

Part (b) – The oscillation frequency

Firstly we make some notational changes, for the sake of convenience. We denote l by the new variable l_0, and let l denote the length of the water column along the plates, at any moment of time. Also, we let 2θ be the angle between the plates at that moment of time.

We introduce a Cartesian coordinate grid with the origin at the hinge and the y axis as the line bisecting the acute angle between the plates, in the considered cross-section. The x axis is the line perpendicular to this. Suppose a point was originally at the coordinates (x_0, y_0), and the point goes to coordinates (x, y) at some time. We establish the relation between (x, y) and (x_0, y_0) first.

We use the hint in this along with the following two paragraphs. The lines $y =$ constant remain horizontal all the time during the small oscillations. Also, the ratio of the x coordinate of a point (x_1, y_1) and the width of the line $y = y_1$ between the plates remains constant, due to the constant density and the fact that the strips along the x axis are always horizontal in their motion. So we have

$$\frac{x}{y\tan\theta} = \frac{x_0}{y_0\tan\theta_0}$$

Also, the area enclosed by a strip and the two plates also remains constant, for it is proportional to the mass of the water below the strip, which is constant as the water in distinct strips doesn't 'mix'. So we have

$$y^2\tan\theta = y_0^2\tan\theta_0$$

Thus we have $(x, y) = \left(x_0\sqrt{\frac{\tan\theta}{\tan\theta_0}}, y_0\sqrt{\frac{\tan\theta_0}{\tan\theta}}\right)$. Also, due to the total conserved mass of the water, or simply from the y coordinate of the highest point, we have $l = l_0\sqrt{\frac{\sin\theta_0\cos\theta_0}{\sin\theta\cos\theta}}$.

Now we find the velocity of each point in the inertial frame where the ceiling is fixed. The signed distance of a particle at the coordinates (x, y) from the ceiling is $-2l_0\cos\theta + y =$

$-2l_0 \cos\theta + y_0\sqrt{\frac{\tan\theta_0}{\tan\theta}}$. Since the y axis remains in the same place due to symmetry, the component of velocity of the particle along the line perpendicular to $\vec{g}$ in the inertial frame is simply

$$v_\perp = \dot{x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(x_0\sqrt{\frac{\tan\theta}{\tan\theta_0}}\right) = \frac{x\dot{\theta}}{2\sin\theta\cos\theta}$$

The component of velocity along $\vec{g}$ is

$$v_\parallel = \frac{\mathrm{d}}{\mathrm{d}t}\left(-2l_0\cos\theta + y_0\sqrt{\frac{\tan\theta_0}{\tan\theta}}\right) = \left(2l_0\sin\theta - \frac{y}{2\sin\theta\cos\theta}\right)\dot{\theta}$$

Thus the kinetic energy of an element of mass $\mathrm{d}m$ at the coordinates (x, y) becomes

$$\mathrm{d}K = \frac{1}{2}\left(v_\parallel^2 + v_\perp^2\right)\cdot \mathrm{d}m = \left(\frac{x^2+y^2}{8\sin^2\theta\cos^2\theta} - \frac{yl_0}{\cos\theta} + 2l_0^2\sin^2\theta\right)\dot{\theta}^2 \cdot \mathrm{d}m$$

The integral of the first term is the moment of inertia of an isosceles triangle of half-vertical angle θ and length of equal sides l, multiplied by $\frac{\dot{\theta}^2}{8\sin^2\theta\cos^2\theta}$, and thus is equal to $\frac{\dot{\theta}^2}{8\sin^2\theta\cos^2\theta}\cdot\frac{1}{2}Ml^2\cdot\left(1-\frac{2}{3}\sin^2\theta\right)$, where M is the mass of the water inside the vessel. The integral of the second term equals y coordinate of the center of mass of the water multiplied by $-\frac{l_0 M\dot{\theta}^2}{\cos\theta}$, so the second term equals $-\frac{l_0 M\dot{\theta}^2}{\cos\theta}\cdot\frac{2}{3}l\cos\theta$. The third term simply is the mass M multiplied by $2l_0^2\sin^2\theta\cdot\dot{\theta}^2$, so it equals $2l_0^2\sin^2\theta\cdot\dot{\theta}^2\cdot M$.
So the net kinetic energy of the system is

$$K = \frac{Ml^2\dot{\theta}^2}{16\sin^2\theta\cos^2\theta}\cdot\left(1-\frac{2}{3}\sin^2\theta\right) - \frac{2Mll_0\dot{\theta}^2}{3} + 2Ml_0^2\sin^2\theta\cdot\dot{\theta}^2 = f(\theta)\dot{\theta}^2$$

The potential energy of the system, with reference point as the ceiling, is given by

$$U = Mg\left(y_{\mathrm{cm}}\right) = Mg\left(-2l_0\cos\theta + \frac{2l_0}{3}\sqrt{\frac{\sin\theta_0\cos\theta_0}{\tan\theta}}\right) = j(\theta)$$

So we have $f(\theta)\dot{\theta}^2 + j(\theta) = \text{constant}$. Differentiating this with respect to time and taking out a factor of $\dot{\theta}$, we have

$$2f(\theta)\ddot{\theta} + \frac{\partial}{\partial\theta}j(\theta) + \frac{\partial}{\partial\theta}f(\theta)\dot{\theta}^2 = 0$$

When we slightly perturb the fluid from equilibrium, the change in potential energy is at most quadratic, so the change in the kinetic energy is also at most quadratic in the coordinate. So the last term in the above equation, for small oscillations, is at most quadratically small in $\Delta\theta = \theta - \theta_0$. Upon expanding the Taylor series of j in $\Delta\theta$, we have

$$j(\theta) = Mg\left(-\sqrt{\frac{40}{27}} + 2\sqrt{\frac{6}{5}}\left(\Delta\theta\right)^2 + \mathcal{O}\left(\left(\Delta\theta\right)^3\right)\right)$$

So indeed, the last term can be safely neglected, and the equation becomes

$$2f(\theta)\ddot{\theta} + \frac{\partial}{\partial\theta}j(\theta) = 0$$

Since $f(\theta_0) \neq 0$, $\ddot{\theta}$ is linear in $\Delta\theta$, so we can safely keep terms only upto zeroth order in $\Delta\theta$ in the coefficient of the first term, for small oscillations. Thus the equation reduces to

$$2f(\theta_0)\ddot{\theta} + \frac{\partial}{\partial\theta}j(\theta) = 0$$

Which on using the mentioned approximations gives that

$$\frac{2Ml_0^2}{15} \cdot \Delta\ddot{\theta} + 4\sqrt{\frac{6}{5}} \cdot Mgl_0 \cdot \Delta\theta = 0$$

This is an equation identical, up to scaling, to that of simple harmonic motion, of the form

$$\ddot{\xi} + \omega^2\xi = 0$$

Comparing this with our equation, we have

$$\omega = \sqrt{\frac{4\sqrt{\frac{6}{5}}Mgl_0}{\frac{2Ml_0^2}{15}}} = \sqrt{30\sqrt{\frac{6}{5}} \cdot \frac{g}{l_0}}$$

In the original notation of the problem, this equals $\sqrt{30\sqrt{\frac{6}{5}} \cdot \frac{g}{l}}$. This is the required oscillation frequency, and the value of the numerical prefactor is 5.733 upto 4 significant digits, after rounding off.