Physics Cup – TalTech 2019 – Problem 4. April 14, 2019

Consider an infinite square grid of resistors. Let us introduce coordinates x and y so that all the nodes are at integer coordinates (n, m), with $n, m \in \mathbb{Z}$. For this grid of resistors, all the horizontal resistors, i.e. the resistors between node pairs $[(n, m), (n + 1, m)]$, have the same resistance R; all the vertical resistors, i.e. the resistors between node pairs $[(n, m), (n, m + 1)]$ have the same resistance r. It appears that for such a grid, the effective resistance R_{nn} between the nodes $(0, 0)$ and (n, n) equals to

$$R_{nn} = \frac{2\sqrt{Rr}}{\pi} \sum_{k=1}^{n} \frac{1}{2k-1};$$

this formula can be used in your solution. By how much will change the effective resistance between the nodes $(0, 0)$ and $(1, 1)$ when the nodes (n, n) and $(n + 1, n + 1)$ are connected with a piece of wire of negligibly small resistance? In other words, determine $R'_{11} - R_{11}$, where R'_{11} is the new effective resistance between the nodes $(0, 0)$ and $(1, 1)$ after short-circuiting the nodes (n, n) and $(n + 1, n + 1)$. Assume that $n > 1$.

Hint 1 (17th Mar. 2019). There are two ways of solving this problem: (a) using an equivalent 4-port circuit made of 6 resistors; (b) making use of the superposition principle — you need to apply it twice. Mathematically, approach (b) is simpler. With (a), the complexity of the algebra depends on the choice of the equivalent circuit (similarly to Δ and Y-connections for 3-port circuits, different circuits can be used).

Hint 2 (24th Mar. 2019). When using approach (a), note that an equivalent circuit needs to have 6 resistors in generic case (this can be proved similarly to how it is shown that a delta or Y-connection can substitute any 3-terminal circuit consisting of resistors). In order to avoid bridge connections, it is convenient to take a circuit with four resistors along the sides of a square $ABCD$, and two resistors dangling from the nodes B and C. The four output nodes of this equivalent circuit are: A and D, together with the dangling ends B' and C'.

When using approach (b), first apply the superposition principle to determine the voltage between (n, n) and $(n+1, n+1)$ when current is driven through $(0, 0)$ and $(1, 1)$. Next, notice that connecting nodes (n, n) and $(n + 1, n + 1)$ with a wire when the nodes (n, n) and $(n + 1, n + 1)$ are connected to a current source is equivalent to drawing a certain unknown current through the nodes (n, n) and $(n + 1, n + 1)$.

Hint 3 (31st Mar. 2019). When using approach (b), in order to determine the voltage between (n, n) and $(n + 1, n + 1)$ while current is driven through $(0, 0)$ and $(1, 1)$, consider the superposition of the following configurations: (A) current I is driven into $(0,0)$, and driven out symmetrically from infinity; (B) current I is driven out from $(1,1)$, and driven in symmetrically at infinity. For the both flows, (A) and (B), the potentials of the nodes (i,i) can be deduced from the effective resistance formula given above.

Hint 4 (7th Apr. 2019). As the final step of the approach (b), consider the superposition of four currents: I driven into $(0,0)$, I driven out from $(0,0)$, I' driven into (n,n), I' driven out from $(n+1,n+1)$; here I' is such a current which compensates the voltage between (n,n) and $(n+1,n+1)$ [explain, why do we need to make the voltage between (n,n) and $(n+1,n+1)$ equal to zero!]

By the end of the fifth week of the fourth problem, there were 405 registered participants from 55 countries; among them there were 204 high school students, and 201 university students. During the three weeks, in total 48 solutions of the fourth problem were submitted, out of which 32 were correct. For the university students, there is still a chance of getting the speed bonus!

Correct solutions submitted by 17th March 2019:

Name	country	Uni/PreUni	subm. date/time (GMT)
Johanes Suhardjo	Indonesia	HKUST	10 Mar. 2019 15:42
Felix Christensen	UK	Oxford	10 Mar. 2019 19:54
Stefan Dolteanu	Romania	PreUni	10 Mar. 2019 20:55
Thomas Foster	UK	Oxford	11 Mar. 2019 10:33
Ionel-Emilian Chiosa	Romania	PreUni	11 Mar. 2019 12:40
Siddharth Tiwary	India	IIT Bombay	11 Mar. 2019 14:34
Tùng Trần Xuân	Vietnam	PreUni	11 Mar. 2019 19:52
Vladislav Polyakov	Russia	PreUni	12 Mar. 2019 06:41
Oliwier Urbański	Poland	PreUni	12 Mar. 2019 14:34
Eduard Burlacu	Romania	PreUni	13 Mar. 2019 16:52
Péter Elek	Hungary	PreUni	13 Mar. 2019 17:41
Mateusz Kapusta	Poland	PreUni	13 Mar. 2019 22:49
Oliver Lindström	Sweden	PreUni	18 Mar. 2019 13:35
Morteza Mudrick	Indonesia	PreUni	20 Mar. 2019 12:09
Ivander J.M. Waskito	Indonesia	PreUni	20 Mar. 2019 13:59
Batuhan Keskin	Turkey	Bogazici Uni	22 Mar. 2019 16:56
Esha Manideep Dinne	India	PreUni	24 Mar. 2019 15:57
Ayush Anand	India	PreUni	24 Mar. 2019 19:22
Kaviraj Prithvi	India	PreUni	25 Mar. 2019 03:24
Vinícius Rodrigues	Brazil	PreUni	01 Apr. 2019 13:54
Masuk Ridwan	Bangladesh	PreUni	06 Apr. 2019 09:16
Stephen Catsamas	Australia	PreUni	06 Apr. 2019 14:04
Dhyaan Nayak	India	PreUni	07 Apr. 2019 12:36
Marvin Janini	Brazil	PreUni	07 Apr. 2019 13:47
Damiano Tietto	Italy	Uni Padova	07 April 2019 16:50
Eugen Dizer	Germany	Uni Heidelberg	07 April 2019 22:55
Jaeyong Lee	Korea	PreUni	08 April 2019 16:37
S.Prajeeth	India	PreUni	11 April 2019 15:35
Roberto Marín Delgado	PreUni	Costa Rica	11 April 2019 18:06
Ričards Kristers Knipšis	PreUni	Latvia	13 April 2019 22:37
Janis Huns	Latvia	PreUni	13 April 2019 13:50