Enrichment in Bicategories

Matt Earnshaw
Tallinn University of Technology

Bob Walters Fest
Tallinn, July 2023

Some history

"1980: I visited Milan and discovered that the sheaf condition can be expressed in terms of Cauchy completeness of categories enriched over a bicategory of 'relations' - this work appeared in Cahiers. A group of us studied categories enriched over bicategories, and bimodules between them. We looked at properties which lifted from the base bicategory to the bicategory of bimodules."

"1982: Lifting the tensor product lead to my idea with Lawvere and Carboni that the classical treatment of this in terms of symmetry could be explained in terms of a tensored one-object bicategory and a Eckmann-Hilton argument - I gave a lecture on this on 26th January 1983 which inspired Ross to discover braided monoidal categories with Joyal (whose motivation was different)."

from *My Interest in Monoidal Bicategories*, 1997
rfcwalters.blogspot.com

Background: Metric spaces as enriched categories

Recall: in enriched categories, homs are no longer sets but objects in some other category.

(Lawvere, 1973) Metric spaces are categories enriched in the poset $\mathcal{V} = ([0, \infty], \geqslant)$ with addition as monoidal product.[1]

- Objects are points, and
- for each pair of objects we have a hom-object $d(x, y) \in [0, \infty]$,
- satisfying $d(x, y) + d(y, z) \geqslant d(x, z)$ (composition), and
- $d(x, x) \geqslant 0$ (units).

$\mathcal{V}$-functors $F : X \to Y$ are contracting maps:

- functions $F : \text{Ob } X \to \text{Ob } Y$,
- satisfying $d_X(x, x') \geqslant d_Y(F(x), F(x'))$ for each pair $x, x' \in X$.

[1] Metric spaces without symmetry and positivity axioms, and with potentially infinite distances.

$\mathcal{V}$-bimodules

We have a tensor product $A \otimes B$ of $\mathcal{V}$-categories:

- Objects are Ob $A \times$ Ob B
- Hom-objects are given by tensor in $\otimes$: $(A \otimes B)((a,b),(a',b')) = A(a,a') \otimes B(b,b')$

For metric spaces, this amounts to the sum of the metrics on the product space: $d_{A\otimes B}((a,b),(a',b')) = d_A(a,a') + d_B(b,b')$.

$\mathcal{V}$-bimodules $P : X \bullet\!\!-\!\!\circ\, Y$ are $\mathcal{V}$-functors $Y^{\mathsf{op}} \otimes X \to \mathcal{V}$ or equivalently:

A family $\{P(y,x)\}_{y \in Y, x \in X}$ of $\mathcal{V}$ objects, equipped with left and right actions:

$$Y(y',y) \otimes P(y,x) \to P(y',x) \qquad P(y,x) \otimes X(x,x') \to P(y,x')$$

$\mathcal{V}$-categories, $\mathcal{V}$-bimodules and $\mathcal{V}$-natural transformations form a bicategory, and thus we have a notion of adjointness for bimodules.

Cauchy-completion for $\mathcal{V}$-categories

(Lawvere, 1973) A metric space Y is Cauchy-complete $\iff$ every pair of adjoint bimodules $p : X \underset{\perp}{\rightleftarrows} Y : q$ is induced by a $\mathcal{V}$-functor $f : X \to Y$.

Suffices to consider points, $X = 1$.

$\mathcal{V}$-functors $1 \to Y$ are just points of Y. Every point $y : 1 \to Y$ gives rise to a pair of adjoint bimodules, "the distance to/from y":

$$y_*(*, x) : Y \multimap 1 := d_Y(y, x), \qquad y^*(x, *) : 1 \multimap Y := d_Y(x, y)$$

Bimodules $1 \multimap Y, Y \multimap 1$ are "virtual points" (decreasing maps $Y^{(\mathsf{op})} \to [0, \infty]$)

Adjointness $p \dashv q : 1 \underset{\perp}{\rightleftarrows} Y$ means:

- Unit: $0 \geqslant \bigwedge_{y \in Y} p(*, y) + q(y, *)$,
- Counit: $q(x, *) + p(*, y) \geqslant d_Y(x, y)$,
- Snake equations (hold automatically)

Cauchy-completion for $\mathcal{V}$-categories cont.

Unit: $0 \geqslant \bigwedge_{y \in Y} p(*, y) + q(y, *)$ Counit: $q(x, *) + p(*, y) \geqslant d_Y(x, y)$

Unit $\Longrightarrow$ for every $n \in \mathbb{N}$ we can choose $y_n \in Y$ s.t. $p(*, y_n) + q(y_n, *) < \frac{1}{n}$.

Now using the counit: let $k, l > n$,
$d_Y(y_k, y_l) \leqslant p(*, y_k) + q(y_l, *) \leqslant p(*, y_k) + q(y_l, *) + p(*, y_k) + q(y_l, *) \leqslant \frac{1}{k} + \frac{1}{l} \leqslant \frac{2}{n}$.

Thus adjoint pairs of bimodules $p : 1 \underset{\perp}{\rightleftarrows} Y : q$ are points in the completion of Y.

Further examples:

- $\mathcal{V} = \mathsf{Set}$, Cauchy-completion is splitting of idempotents.
- For a ring R considered as a $\mathcal{V} = \mathsf{AbGrp}$ enriched category, Cauchy-completion is the category of finitely generated projective R-modules.

$\mathcal{V}$-categories as lax 2-functors, and $\mathcal{W}$-categories

Consider the monoidal $\mathcal{V}$ as a one-object bicategory $\Sigma\mathcal{V}$: morphisms are the objects of $\mathcal{V}$, 2-cells the morphisms of $\mathcal{V}$, composition $\otimes$.

For a set of objects X, let X_{ch} be the "chaotic" bicategory: objects X and trivial hom-categories.

Then the data of a $\mathcal{V}$-category with objects X is exactly that of a lax 2-functor $X_{\mathsf{ch}} \to \Sigma\mathcal{V}$.

The local functors $F_{A,B} : 1 \to \Sigma\mathcal{V}(\bullet, \bullet)$ pick out the hom-object from A to B.

Laxators give us identities and composition:

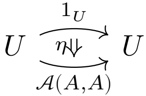
- for each $A \in X$, a 2-cell $\mathsf{Id}_\bullet \to F_{A,A}(\mathsf{Id}_A)$,
- for each triple $A, B, C \in X$, a 2-cell $F_{B,C}(1) \circ F_{A,B}(1) \to F_{A,C}(1 \circ 1)$.

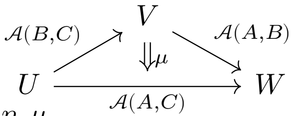
Replacing $\Sigma\mathcal{V}$ with an arbitrary bicategory $\mathcal{W}$, we obtain categories enriched in $\mathcal{W}$.[2]

[2] Replacing the functor with certain spans of functors gives "categories enriched on two sides" (Kelly, Labella, Schmitt, Street).

$\mathcal{W}$-categories

More explicitly, a $\mathcal{W}$-category $\mathcal{A}$ is given by

- For each $U \in \mathcal{W}$, a set of objects $\mathcal{A}_U$ over U, (for $x \in \mathcal{A}_U$ write $e(x) = U$)
- for objects A, B over U, V respectively, a 1-cell $\mathcal{A}(A, B) : U \to V$ in $\mathcal{W}$,
- for objects A, B, C over U, V, W, 2-cells: 



- pasting diagrams expressing unitality and associativity of η, μ.

(Walters, 1981) "Draw a picture. $\mathcal{A}$ is a space lying over $\mathcal{W}$."

This idea was in the air: in notes of (Betti, 1980), but also (Bénabou, 1967) had called these *polyads*, since monads in $\mathcal{W}$ are the case where $\mathcal{A} = 1$.

Walters' first advance was to provide a serious example (particularly of Cauchy-completion). Motivated by this, he went on to deepen the theory.

$\mathcal{W}$-functors and $\mathcal{W}$-bimodules / -modules / -profunctors / -distributors

A $\mathcal{W}$-functor $F : \mathcal{A} \to \mathcal{B}$ is given by:

- A function $F : \text{Ob } A \to \text{Ob } B$, such that for $A \in \mathcal{A}_U \Rightarrow FA \in \mathcal{B}_U$, and
- a 2-cell $U \overset{\mathcal{A}(A,A')}{\underset{\mathcal{B}(FA,FA')}{\Downarrow}} V$ in $\mathcal{W}$ for each pair $A, A' \in \mathcal{A}$ over U, V.

Bimodules now given by indexed family of 1-cells in $\mathcal{W}$ equipped with actions.

For $\mathcal{W}$-categories $\mathcal{A}, \mathcal{B}$, a bimodule $\Phi : \mathcal{A} \bullet\!\!-\!\!\circ \mathcal{B}$ is given by:

- a 1-cell $\Phi(B, A) : V \to U$ in $\mathcal{W}$, for each pair $A \in \mathcal{A}, B \in \mathcal{B}$ over U, V respectively,
- a 2-cell $r : \Phi(B, A) \circ \mathcal{A}(A', A) \to \Phi(B, A)$, for each pair $A, A' \in \mathcal{A}$ and $B \in \mathcal{B}$,
- a 2-cell $\ell : \mathcal{B}(B', B) \circ \Phi(B, A) \to \Phi(B', A)$, for each $A \in \mathcal{A}$ and pair $B, B' \in \mathcal{B}$.

satisfying axioms making r, ℓ into (compatible) actions.

Adjoint $\mathcal{W}$-bimodules

At first Walters only considered bicategories whose hom-categories are posets, in which case the axioms for 2-cells hold automatically.

Composition of bimodules can be defined as a colimit, we will only need posetal case.

For locally posetal $\mathcal{W}$-categories, bimodules $p : \mathcal{A} \underset{\perp}{\rightleftarrows} \mathcal{B} : q$ are adjoint if:

- $\mathcal{A}(A, A') \leqslant \bigvee_B p(B, A) \circ q(A', B)$ (sup in the appropriate hom-poset)
- $\bigvee_A q(A, B) \circ p(B', A) \leqslant \mathcal{B}(B, B')$

Cauchy-complete $\mathcal{W}$-categories are then defined exactly as for $\mathcal{V}$-categories.

Sheaves as sets with equality in a locale

(Higgs, 1973) and (Fourman and Scott, 1979) developed a perspective on sheaves (on locales) as sets with a locale-valued equality.

- A a set, $\mathcal{O}(X)$ a locale,
- $[\bullet \simeq \bullet] : A \times A \to \mathcal{O}(X)$, a function such that
- $[a \simeq b] = [b \simeq a]$, and
- $[b \simeq c] \wedge [a \simeq b] \leqslant [a \simeq c]$.

There is an equivalence of categories between $O(X)$-sets and sheaves on X.

This looks like a category enriched in the locale, but lacking identities.

Walters' insight: by constructing an appropriate bicategory from the locale, we can refine the base of enrichment and get an exact correspondence.

Presheaves on locales as $\mathcal{W}$-categories

Given a locale $\mathcal{O}(X)$ we form the bicategory $\mathsf{Rel}(\mathcal{O}(X))$:

- 0-cells elements $U \in \mathcal{O}(X)$
- 1-cells $U \to V$ are elements $W \subseteq U \wedge V$
- 2-cells given by $\subseteq$ in $\mathcal{O}(X)$
- Composition given by $\wedge$ in $\mathcal{O}(X)$

Given a presheaf $F : \mathcal{O}(X)^{\mathsf{op}} \to \mathsf{Set}$, we can form a $\mathsf{Rel}(\mathcal{O}(X))$-category $\mathcal{F}$:

Take the objects over U to be the set $F(U)$.

Take as hom $\mathcal{F}(s \in F(U), t \in F(V))$ the largest of those $W \subseteq U \wedge V$ where the restrictions $s|_W = t|_W$ agree.

(Walters, 1981) F is a sheaf precisely when $\mathcal{F}$ is Cauchy-complete.

Sheaf condition as Cauchy-completion

(Walters, 1981) F is a sheaf precisely when $\mathcal{F}$ is Cauchy-complete.

A presheaf on a locale is a *sheaf* when "every compatible family glues uniquely":

Given $U \subseteq \bigvee_i U_i$, and a family of sections $\{x_i \in F(U_i)\}$ compatible in the sense that $x_i|_{U_i \wedge U_j} = x_j|_{U_i \wedge U_j}$, then there exists a unique $x \in F(U)$ such that $x|_{U_i} = x_i$.

For each $U \in \mathcal{W}$ there is a one-object $\mathcal{W}$-category $\hat{U}$ with $*$ over U and $\hat{U}(*,*) = 1_U$.

Suffices to consider modules to/from $X = \hat{U}$ for all $U \in \mathcal{W}$.

A $\mathcal{W}$-functor $s : \hat{U} \to \mathcal{F}$ is a section $s \in F(U)$. Every section $s \in F(U)$ gives rise to a pair of adjoint bimodules, assigning to each section $t \in F(V)$ the largest $W \subseteq U \wedge V$ such that $s|_W = t|_W$.

$$s_*(*,t) : \mathcal{F} \bullet\!\!-\!\!\circ \hat{U} := \mathcal{F}(s,t), \qquad s^*(t,*) : \hat{U} \bullet\!\!-\!\!\circ \mathcal{F} := \mathcal{F}(t,s)$$

Sheaf condition as Cauchy-completion (cont.)

Now consider adjoint bimodules $p \dashv q : \hat{U} \nrightarrow \mathcal{F}$. Adjointness means:

- Unit: $U \subseteq \bigvee_{s \in \mathcal{F}} (p(*, s) \wedge q(s, *))$, so $\{U_s := p(*, s) \wedge q(s, *)\}_s$ covers U
- Counit: $q(s, *) \wedge p(*, t) \subseteq \mathcal{F}(s, t)$

Counit implies $U_s \wedge U_t \subseteq \mathcal{F}(s, t)$, so $\{s|_{U_s}\}_s$ is a compatible family.

F is a sheaf $\Rightarrow$ there exists a unique $s_0 \in F(U)$ such that $s_0|_{U_s} = s|_{U_s}$.

Claim: $s_0 : \hat{U} \to \mathcal{F}$ represents the adjoint pair, $p(*, s) = \mathcal{F}(s_0, s) = q(s, *)$.

Follows from unit/counit and properties of bimodules.

(Walters, 1981) The category of sheaves on $\mathcal{O}(X)$ is equivalent to the category of skeletal symmetric Cauchy-complete Rel($\mathcal{O}(X)$)-categories.

(Walters, 1982) Generalizes this to sheaves on arbitrary sites. In this case the enrichment is in a bicategory of relations not arising as internal relations.

A contemporary perspective

An indexed family of monoidal categories $F : \mathcal{C}^{\mathsf{op}} \xrightarrow{psd.} \mathsf{MonCat}$ is equivalently a monoidal fibration $\int F \to \mathcal{C}$, when $\mathcal{C}$ is cartesian monoidal.

(Shulman, 2007) shows that monoidal fibrations give rise to *framed bicategories* (double categories with extra properties).

For a locale $\mathcal{O}(X)$, define $\mathcal{S} : \mathcal{O}(X)^{\mathsf{op}} \to \mathsf{MonCat}$ by:

- mapping an open U to the monoidal poset $(\otimes = \wedge)$ of opens $V \subseteq U$, and
- for each $U \subseteq V$, define the monoidal functor $\mathcal{S}(V) \to \mathcal{S}(U) : W \mapsto W \cap U$.

The resulting framed bicategory has loose bicategory that of (Walters, 1981).

We can enrich in double categories: just take the underlying loose bicategory.

What is different is the resulting wider notion of enriched functor, which is the correct one in many cases. The requirement that when A lives over U then FA also lives over U, can be relaxed by requiring a compatible family of tight morphisms $U \to V$.

Bibliography I: Some work by others stemming from these ideas

- (1981) Street, *Cauchy characterization of enriched categories*
  » Early characterization of bicategories biequivalent to $\mathcal{W}$-Mod.
- (1982) Betti, Carboni, *Cauchy-completion and the associated sheaf*
  » Proving that Cauchy-completion always exists for $\mathcal{W}$-categories, and further analysis of sheafification.
- (1983) Street, *Enriched categories and cohomology*
  » Extension to stacks, with applications to torsors and cohomology.
- (1984) Betti, Kasangian, *Tree automata and enriched category theory*
  » Encoding tree automata as categories enriched in the free quantaloid over a Lawvere theory.
- (1992) Verity, *Enriched categories, internal categories and change of base*
  » Axiomatizes generalized sites as bicategories with certain exactness properties, amongst other things.
- (1997) Gordon, Power, *Enrichment through variation*
  » Generalizes Gabriel-Ulmer duality to $\mathcal{W}$-categories.
- (1999) Leinster, *Generalized enrichment for categories and multicategories*
  » Enrichment in virtual double categories (fc-multicategories) as unifying definition.
- (2004) Stubbe, *Categorical structures enriched in a quantaloid*
  » Extended consideration of the case of quantaloids, with applications.
- (2006) Schmitt, Worytkiewicz, *Bisimulation of enrichments*
  » Lifting bisimulation to $\mathcal{W}$-categories.
- (2012) Cockett, Garner, *Restriction categories as enriched categories*
  » Restriction categories as categories enriched in a weak double category.
- (2013) Garner, Shulman, *Enriched categories as a free cocompletion*
  » Develops the theory of bicategories enriched in monoidal bicategories, exhibiting $\mathcal{W} \mapsto \mathcal{W}$-Cat as the free cocompletion of an enriched bicategory.

… and much more.

Bibliography II: Walters' work

- (1981) *Sheaves and Cauchy complete categories*
  » Category of sheaves on a locale H equivalent to category of skeletal symmetric Cauchy complete $\mathcal{W}(H)$-categories.
- (1981) *The symmetry of the Cauchy completion of a category* (with R. Betti)
  » If the base bicategory satisfies the modular law, then symmetry is preserved by Cauchy completion.
- (1982) *Sheaves on sites as Cauchy-complete categories*
  » Extension of the first paper to sheaves for arbitrary Grothendieck topologies.
- (1982) *Variation through enrichment* (with R. Betti, A. Carboni & R. Street)
  » Colimits of $\mathcal{W}$-categories and fibrations as $\mathcal{W}$-categories.
- (1983) *On the completeness of locally internal categories* (with R. Betti)
  » Treats the theory of locally internal categories by considering them as enriched in Span$(\mathcal{E})$, for $\mathcal{E}$ a topos.
- (1985) *Closed bicategories and variable category theory* (with R. Betti)
  » Report on talks at Sydney CT Seminar. More work on locally internal categories as enrichment in Span.
- (1985) *An axiomatics for bicategories of modules* (with A. Carboni & S. Kasangian)
  » Proof that $\mathcal{W} \mapsto \mathcal{W}$-Mod is idempotent, leading to a characterization of categories of modules.
- (1989) *The calculus of ends over a base topos* (with R. Betti)
  » Further work on locally internal categories as enriched categories, developing a calculus of ends.
- (1994) *Representations of modules and Cauchy completeness* (with Shu Hao Sun)
  » Initiated the analysis of categories of modules over rings as categories enriched over various bases.