A short note on totally periodic configurations

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Suppose $c : \mathbb{Z}^d \to S$ is totally periodic. This means that there are d linearly independent vectors $\vec{r}_i$ with integer coordinates such that $c(\vec{v} + \vec{r}_i) = c(\vec{v})$ for every $\vec{v} \in \mathbb{Z}^d$ and every $i \in \{1, \ldots, d\}$.

Consider the matrix A whose i-th column is the vector $\vec{r}_i$: that is, $A_{j,i}$ is the j-th coordinate of $\vec{r}_i$ according to the base $\{\vec{e}_1, \ldots, \vec{e}_d\}$. Then A^{-1} is the matrix whose i-th column contains the coordinates of the vector $\vec{e}_i$, expressed as a linear combination of the elements of the base $\{\vec{r}_1, \ldots, \vec{r}_d\}$: that is, $\vec{e}_i = A^{-1}_{1,i}\vec{r}_1 + \ldots + A^{-1}_{d,i}\vec{r}_d$.

But A is a matrix with integer components: then A^{-1} is a matrix with *rational* components, as its elements are determinants of matrices with integers components (the minors of A) divided by an integer value (the determinant of A). If n is such that the components of nA^{-1} are all integers (for example, n is a multiple of all the denominators in the writings of the elements of A^{-1} as irreducible fractions) then c is $n\vec{e}_i$-periodic for every $i \in \{1, \ldots, d\}$.

For example, consider a situation where the basic period is a von Neumann neighborhood. The configuration c in Figure 1 is $\vec{r}$-periodic for $\vec{r} = \vec{r}_1 = (2, 1)$ and for $\vec{r} = \vec{r}_2 = (-1, 2)$: these vectors are linearly independent, as the matrix $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ whose i-th column is $\vec{r}_i$ has determinant $2 \cdot 2 - (-1) \cdot 1 = 5$. The inverse matrix of A is $A^{-1} = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$: thus, $\vec{e}_1 = \frac{2}{5}\vec{r}_1 - \frac{1}{5}\vec{r}_2$ and $\vec{e}_2 = \frac{1}{5}\vec{r}_1 + \frac{2}{5}\vec{r}_2$. Then c is $\vec{r}$-periodic also for $\vec{r} = (5, 0) = 2\vec{r}_1 - \vec{r}_2$ and for $\vec{r} = (0, 5) = \vec{r}_1 + 2\vec{r}_2$.

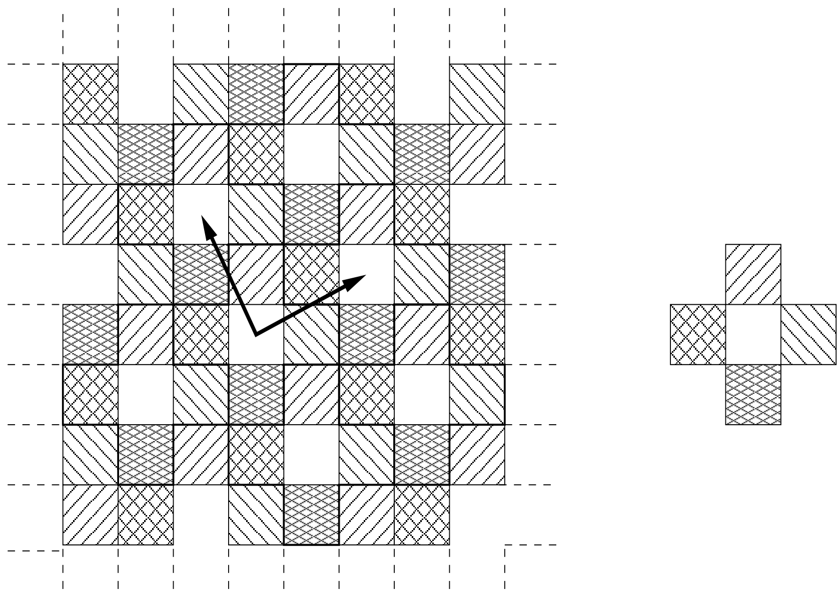

Figure 1: An example of a periodic configuration whose basic period is not a square.