ITT8040 — Cellular Automata
Lecture 3

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Patterns

A d-dimensional pattern over S is a pair $p = (D, G)$ where:

- $D \subseteq Z^d$ is the domain, and
- $g : D \to S$.

For $\vec{r} \in \mathbb{Z}^d$ we put

$$\tau_{\vec{r}}(p) = (D + \vec{r}, g \circ \tau_{-\vec{r}})$$

$p_1 = (D_1, g_1)$ is a subpattern of $p_2 = (D_2, g_2)$ if

- $D_1 \subseteq D_2$, and
- $g_2|_{D_1} = g_1$.

Cellular automata and patterns

Let (S, d, N, f) be a cellular automaton with global transition function G.

- Let $D, D' \subseteq \mathbb{Z}^d$ such that $N(D') \subseteq D$.
- We define a function $G^{(D \to D')} : S^D \to S^{D'}$ as follows: If $p = (D, G)$ then $G^{(D \to D')}(p) = (D', g')$ where

$$g'(\vec{n}) = f\left(g(\vec{n} + \vec{n}_1), \ldots, g(\vec{n} + \vec{n}_m)\right) \quad \forall \vec{n} \in D'$$

Observe that this is the same as doing the following:

1. First, extend $g : D \to S$ to some $c : \mathbb{Z}^d \to S$.
2. Next, compute $G(c)$.
3. Finally, set $g' = G(c)|_{D'}$.

Call orphan a pattern that has no preimage.

The orphan pattern principle

Let $A = (S, d, N, f)$ be a cellular automaton. The following are equivalent:

1. A has a Garden-of-Eden configuration.
2. A has an orphan pattern.

Proof: By compactness of $S^{\mathbb{Z}^d}$ and continuity of the global function G.

Balancedness

Let (S, d, N, f) be a cellular automaton with global transition function G.

- Let D, D' be finite subsets of $\mathbb{Z}^d$ be such that $N(D') \subseteq D$.
- We say that the CA is D'-balanced if for every pattern $p' = (D', g')$,

$$\left| \left(G^{(D \to D')} \right)^{-1} (p') \right| = |S|^{|D|-|D'|}$$

- We say that the CA is balanced if it is D'-balanced for every finite $D' \subseteq \mathbb{Z}^d$.

Note: A balanced CA is surjective.

The balancedness theorem (Maruoka and Kimura, 1976)

- Let (S, d, N, f) be a surjective cellular automaton with global transition function G.
- Let D, D' be finite subsets of $\mathbb{Z}^d$ such that $N(D') \subseteq D$.
- Then, for every d-dimensional pattern $p' = (D', g')$ over S,

$$\left| \left(G^{(D \to D')} \right)^{-1} (p') \right| = |S|^{|D| - |D'|},$$

that is, there are exactly $|S|^{|D|-|D'|}$ patterns $p = (D, g)$ such that $G^{(D \to D')}(p) = p'$.

Shortly:

a surjective CA is balanced

Proof of the balancedness theorem

It is not restrictive to only consider hypercubic supports.

- Suppose p' has $t \neq |S|^{|D|-|D'|}$ preimages.
- Take E, E' hypercubes with $D \subseteq E$, $D' \subseteq E'$, and $N(E') \subseteq E$.
- Exactly $|S|^{|E'|-|D'|}$ patterns on E' have p' as subpattern.
- If the CA was E'-balanced, then each of those would have exactly $|S|^{|E|-|E'|}$ preimages.
- But those patterns have $t \cdot |S|^{|E|-|D|}$ possible preimages by $G^{(E \to E')}$: thus,

$$|S|^{|E'|-|D'|} \cdot |S|^{|E|-|E'|} = t \cdot |S|^{|E|-|D|}$$

- This is impossible, at it implies $t = |S|^{|D|-|D'|}$.

Proof of the balancedness theorem (cont.)

Moore's inequality: Let d, n, s, r be positive integers. For every k large enough,

$$\left(s^{n^d} - 1\right)^{k^d} < s^{(kn-2r)^d}$$

Suppose now the CA is not balanced. We set:

- D a hypercube of side n,
- D' a hypercube of side $n - 2r$ with $N \subseteq \{-r, \ldots, r\}^d$, and
- $p' = (D', g')$ a pattern with $t < |S|^{|D|-|D'|}$ preimages.

Then $|D| = n^d$ and $|D'| = (n - 2r)^d$ ...

Proof of the balancedness theorem (end.)

Let now D_k be a hypercube of side kn, and D'_k the hypercube of side $kn - 2r$ centered in its middle.

- D_k is made of k^d hypercubes of side n. D'_k is a hypercube of side $kn - 2r$.
- There are $|S|^{|D'_k| - k^d \cdot |D'|}$ patterns over D'_k that coincide with p' in correspondence of the centers of the k^d hypercubes above.
- Then there are at most t^{k^d} preimages for such patterns: but

$$t^{k^d} \leq \left(|S|^{|D| - |D'|} - 1 \right)^{k^d} \leq |S|^{-k^d |D'|} \left(|S|^{|D|} - 1 \right)^{k^d}$$

- By Moore's inequality with $s = |S|$, $t^{k^d} < |S|^{|D'_k| - k^d |D'|}$ for k large enough. For such k, some patterns on D'_k are orphans.

Pre-injectivity

Two configurations $c, e \in S^{\mathbb{Z}^d}$ are asymptotic if the set

$$\Delta(c, e) = \{\vec{n} \in \mathbb{Z}^d \mid c(\vec{n}) \neq e(\vec{n})\}$$

is finite.
A cellular automaton with global function G is pre-injective if $G(c) \neq G(e)$ whenever c and e are different and asymptotic.
Shortly:

a cellular automaton is pre-injective
if it cannot correct finitely many errors in finite time

This is actually the same as saying that the CA is injective in

$$\mathrm{asymp}(c) = \{e \in S^{\mathbb{Z}^d} \mid |\Delta(c, e)| < \infty\}$$

where c is any given configuration. In particular:

if G has a quiescent state,
then G is pre-injective if and only if G_F is injective

Theorem (Myhill, 1962)

If a CA has a Garden-of-Eden configuration,
then it is not pre-injective.

Proof of Myhill's theorem

Let (S, d, N, f) be a non-surjective CA with global function G. Suppose $N = \{-r, \ldots, r\}^d$.

- Fix $q \in S$ and set $t = f(q, \ldots, q)$.
- Let p be a GoE pattern. Pad p with t to form a side-n hypercube.
- Let C be a side-kn hypercube and C' the side-$(kn - 2r)$ hypercube centered in the middle of C.
- Let $K = \{c \in S^{\mathbb{Z}^d} \mid c(\vec{n}) = q \, \forall \vec{n} \notin C'\}$. Then $|K| = |S|^{(kn-2r)^d}$. Also, if $c \in K$ then $(G(c))(\vec{n}) = t$ for every $\vec{n} \notin C$.
- But there are at most $\left(|S|^{n^d} - 1\right)^{k^d}$ possible choices for $G(c)$.

Consequences of Myhill's theorem

- If G_F is injective then G is surjective.
- If G is injective then G is surjective.
- If G is injective then G_F is surjective.

The Garden-of-Eden theorem, part 2

Theorem (Moore, 1962)

If a CA is not pre-injective,
then it has a Garden-of-Eden configuration.

Proof of Moore's theorem

Let (S, d, N, f) have $N = \{-r/2, \ldots, r/2\}^d$ and global function G. Let $c_1, c_2 : \mathbb{Z}^d \to S$ be asymptotic. Suppose $G(c_1) = G(c_2)$.

- Let D' be an hypercube of side $n - 2r$ so large that $\Delta(c_1, c_2) \subseteq D'$. Let D be the hypercube of side n with same center as D'. Call $p_i = (D, c_i|_D)$.
- Then in any configuration c, any copy of p_1 can be replaced by a copy of p_2 and vice versa, without affecting $G(c)$.
- Let now D_k have side kn, and let D'_k be the side-$(kn - 2r)$ hypercube with same center as D_k.
- Then there are at most $\left(|S|^{n^d} - 1\right)^{k^d}$ available preimages for $|S|^{(kn-2r)^d}$ patterns.