ITT8040 — Cellular Automata
Lecture 8

Silvio Capobianco

Institute of Cybernetics at TUT

April 29, 2013

Undecidable problems about cellular automata

- Nilpotency.
- Surjectivity in dimension 2 (or greater).
- **Reversibility in dimension 2 (or greater).**

How can we ensure reversibility in CA?

Lattice gas cellular automata

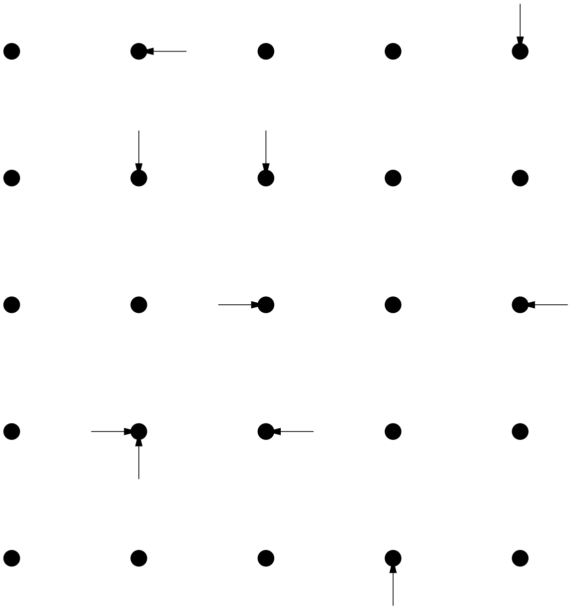
A lattice gas cellular automaton (LGCA) is a CA (S, d, N, f) where:

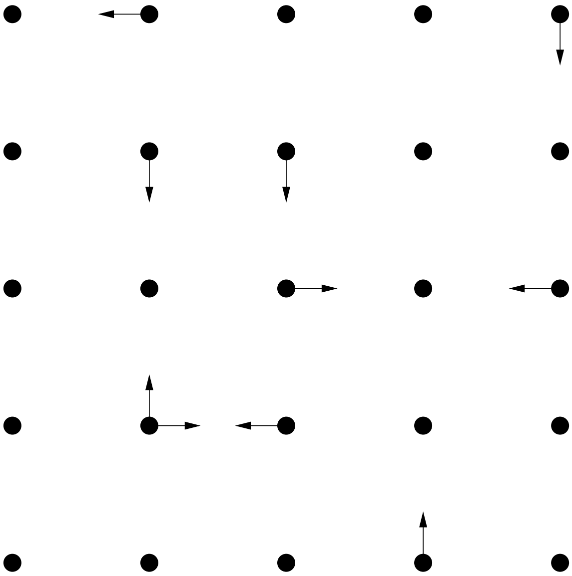
- The set of states $S = S_1 \times \ldots \times S_m$ has as many channels as the elements of $N = \{\vec{n}_1, \ldots, \vec{n}_m\}$.
- The global evolution function is a sequence of two steps:
  1. Propagation:
     each track S_i is shifted by $\vec{n}_i$;
  2. Interaction:
     a transformation $\pi : S \to S$ is performed.

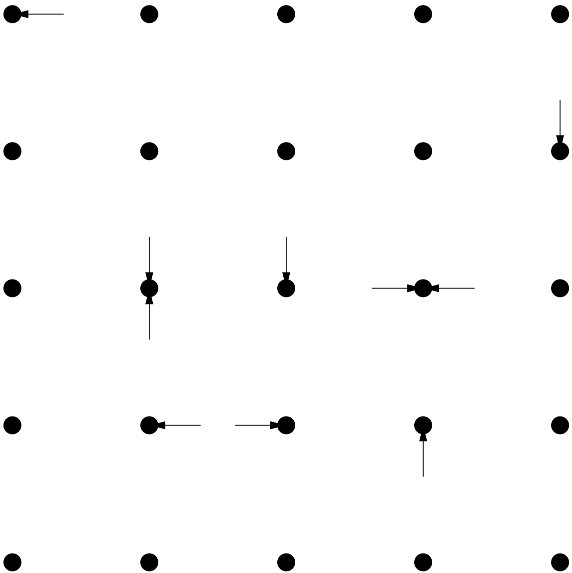
HPP: the first LGCA

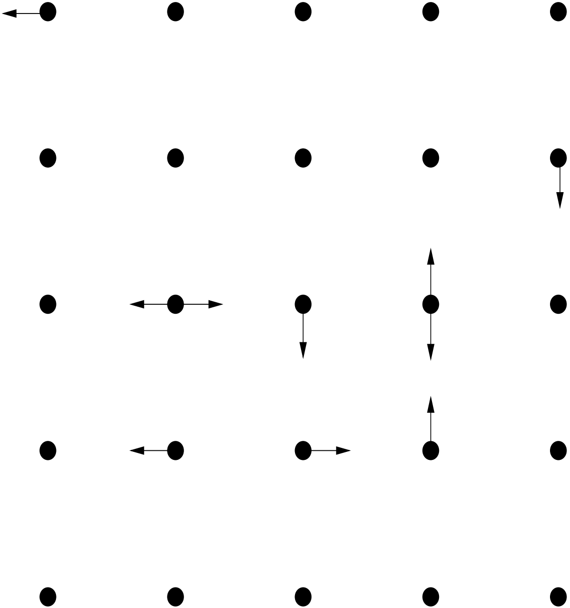
Hardy, de Pazzis and Pomeau

- Square grid, four directions.
- Particle moving along channels between nodes of the grid.
- Propagation: to nearest neighbor.
- Interaction:
  - If exactly two particles arrive from opposite directions, then they bounce at a right angle.
  - In all other cases, the particles go straight ahead.









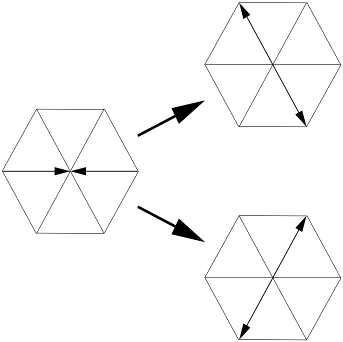
FHP: a better lattice gas model

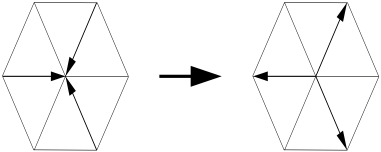
HPP has several problems, which make it unsuited as a gas model:

- Lattice directions are privileged.
- Horizontal momentum is preserved along horizontal lines. Such conservation law does not hold for real gasses.

Frisch, Hasslacher and and Pomeau

- Triangular grid, six directions.
- Particle moving along channels between nodes of the grid.
- Propagation: to nearest neighbor.
- Interaction:
  - If exactly two particles arrive from opposite directions, then they bounce by 60 degrees at a random direction.
  - If three particles arrive by 120 degrees from each other, then they bounce away.
  - In all other cases, the particles go straight ahead.





Reversibility of LGCA

The following are equivalent:

1. The LGCA is reversible.
2. The global function of the LGCA interaction phase is bijective.
3. The local function of the LGCA interaction phase is a permutation.

As a side effect:

if a LGCA is not injective,
then it is not surjective either

Partitioned cellular automata

A partitioned cellular automaton is a CA (S, d, N, f) where:

- The set of states $S = S_1 \times \ldots \times S_m$ has as many tracks as the elements of $N = \{\vec{n}_1, \ldots, \vec{n}_m\}$.
- The global evolution function is a sequence of two steps:
  1. first, each track S_i is shifted by $\vec{n}_i$;
  2. next, a permutation $\pi : S \to S$ is performed;

  that is, G is a composition of partial shifts and a point-based permutation.

Universality of RPCA

Theorem

There exists an algorithm that, given in input an arbitrary Turing machine, returns a reversible one-dimensional partitioned cellular automaton that simulates the Turing machine in real time.

Corollary

There exists a reversible 1D PCA and a subset F of its states for which the following problem is r.e.-complete:

given a finite configuration c,
determine whether it ultimately evolves into a configuration
which has some of its states in F.

Simulating TM by RPCA: the main issue

Reversibility means that no information is erased.

- We need a "garbage track" to take care of previous states.

Let the Turing machine $M = (Q, q_I, q_A, q_R, \Gamma, \Sigma, \delta)$ be given.
Set $S = S_1 \times S_2 \times S_3 \times S_4$ with:

- $S_1 = \Gamma$.
- $S_2 = Q \sqcup \{0\}$.
- $S_3 = Q \sqcup \{0\}$.
  Until now, the construction is the same as for the non-reversible case, except that we use two tracks to encode motion of the read-write head.
- $S_4 = (Q \times \Gamma \times \{-1, +1\}) \sqcup \{0\}$.
  This is the "garbage track", which will be moved two slots per time unit, so that it does not interfere with the computation.

Simulating TM by RPCA: the local function

Set $N = \{0, -1, +1, +2\}$.

- Track 1 does not move.
- Track 2 is shifted one position to the left.
- Track 3 is shifted one position to the right.
- Track 4 is shifted two positions to the right.

Define $\pi : S \to S$ as a permutation that fulfills the conditions:

- if $\delta(q, a) = (q', a', -1)$
  then $(a, q, 0, 0) \mapsto (a', q', 0, (q, a, -1))$
  and $(a, 0, q, 0) \mapsto (a', q', 0, (q, a, +1))$;
- if $\delta(q, a) = (q', a', +1)$
  then $(a, q, 0, 0) \mapsto (a', 0, q', (q, a, -1))$
  and $(a, 0, q, 0) \mapsto (a', 0, q', (q, a, +1))$;
- $(a, 0, 0, g) \mapsto (a, 0, 0, g)$ whatever g is.

Reversible CA as PCA

Kari, 1996

- No additional state
- Proved in dimension 1 and 2
- Conjectured for higher dimension

Durand-Lose, 1999

- Additional state
- Worsk in arbitrary dimension

Higher block presentations

Let $(S, 1, N, f)$ be a one-dimensional CA.
The m-block presentation of the CA is determined by the block merging function

$$B_m : S^{\mathbb{Z}} \to (S^m) \to \mathbb{Z} \text{ such that } B_m(c)(i) = c_{[mi, mi+m-1]}$$

and the block splitting function

$$B_m^{-1} : (S^m) \to \mathbb{Z} \to S^{\mathbb{Z}} \text{ such that } B_m^{-1}(e)(i) = e(\lfloor i/m \rfloor)(i \bmod m)$$

so that the global function of the m-th higher block presentation is

$$B_m \circ G \circ B_m^{-1}$$

A similar idea works in higher dimension.

Reversible 1D CA as PCA: construction

Let G be the global function of a reversible 1D CA.
Suppose G and G^{-1} are both defined by a radius-r neighborhood.
Set $n = 3r$. Define the set of right stairs:

$$\mathcal{R} = \{(c_{[0,2r-1]}, G(c)_{[-r,r-1]}) \mid c \in S^{\mathbb{Z}}\} \subseteq^{2r} \times S^{2r}$$

and the set of left stairs:

$$\mathcal{L} = \{(G(c)_{[0,2r-1]}, c_{[-r,r-1]}) \mid c \in S^{\mathbb{Z}}\} \subseteq S^{2r} \times S^{2r}$$

The function $\phi : S^{6r} \to \mathcal{R} \times \mathcal{L}$ defined by

$$\phi(c_0, \ldots, c_{6r-1}) = ((c_{[4r,6r-1]}, G(c)_{[3r,5r-1]}), (G(c)_{[r,3r-1]}, c_{[0,2r-1]}))$$

is a bijection, and so is $\psi : \mathcal{R} \times \mathcal{L} \to S^{6r}$ defined by

$$\psi((c_{[4r,6r-1]}, G(c)_{[3r,5r-1]}), (G(c)_{[r,3r-1]}, c_{[0,2r-1]})) = G(c)_{[0,6r-1]}$$

Reversible 1D CA as PCA: representation

Consider the radius-1/2 PCA with set of states $\mathcal{R} \times \mathcal{L}$ and permutation function

$$\pi = \psi \circ \phi$$

Such a PCA is isomorphic to the $6r$-block presentation of $G \circ \sigma^{3r}$.