ITT8040 — Cellular Automata
Lecture 9

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May 8, 2013

Block cellular automata

A block cellular automaton with block sides $m_1, \ldots, m_d$ is composed of two phases:

1. a translation, and
2. a block transformation $b : S^m \to S^m$, where $m = m_1 \cdots m_d$.

The global update is performed as follows:

1. The space is partitioned into hypercubic blocks of sides $m_1, \ldots, m_d$.
2. The transformation b is applied to each block.
3. The translation is performed.
   (This step corresponds to a change in the origin of the partitioning.)

Reversibility of block CA

As it is the case with LGCA and partitioned CA, there are only two kinds of block CA:

1. reversible block CA, if b is a permutation;
2. non-surjective, non-injective block CA otherwise.

Subadditivity

A function $f : \{1, 2, \ldots\} \to \mathbb{R}$ is subadditive if for every $n, m > 0$

$$f(n+m) \leq f(n) + f(m)$$

Lemma (Fekete)
If f is a subadditive function, then

$$\lim_{n \to \infty} \frac{f(n)}{n} = \inf_{n \geq 1} \frac{f(n)}{n}$$

Variety

Let A be a one-dimensional CA.
Call $\mathrm{Out}(n)$ be the number of possible states of the interval $\prod_{i=1}^{d}\{1,\ldots,n\}$ after an application of A's global function.
The variety of A is the function

$$V(n) = \log_{|S|} \mathrm{Out}(n)$$

Exactly one of the following happens:

1. A is surjective and $V(n) = n$ for every n.
2. A is non-surjective and for every $k > 0$ there exists n_k such that $V(n) < n - k$ for every $n > n_k$.

Representing non-surjective 1D CA as block CA

Theorem (Toffoli, Capobianco and Mentrasti, 2008)
Every non-surjective 1D CA can be rewritten as a two-layer block CA.

- Let m be the neighborhood range.
- Suppose n is so large that $V(n) < n - m$.
- Consider blocks of size $n + m$.
- Let the first block operation compress the output of the n leftmost sites into $n - m$ values, and leave the m rightmost unchanged.
- Shift by m units to the left.
- Let the second block operation decompress the n rightmost values and use them to compute the next state of the m leftmost sites.
- Shift by m units to the right.

Second-order cellular automata

A second-order cellular automaton is a CA (S, d, N, f) where $f : S^m \times S$, that is, where the equation of the orbits has the form:

$$c^{t+1}(\vec{n}) = f(c^t(\vec{n} + \vec{n}_1), \ldots, c^t(\vec{n} + \vec{n}_m); c^{t-1}(\vec{n}))$$

The next configuration is thus determined by both the current and the previous ones. The global function of a second-order CA has thus the type:

$$G : S^{\mathbb{Z}^d} \times S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$$

and the equation of the second-order dynamics is:

$$c^{t+1} = G(c^t; c^{t-1})$$

Reversibility for second-order CA

A second-order CA is reversible if there exists a second-order CA whose global function H satisfies the reverse-time equation, that is,

$$\forall c^t, c^{t-1} \in S^{\mathbb{Z}^d} : \ c^{t+1} = G(c^t; c^{t-1}) \Rightarrow c^{t-1} = H(c^t; c^{t+1})$$

For a second-order CA with local update rule f and global function G, the following are equivalent:

1. The second-order CA is reversible.
2. For every $k \in S^{\mathbb{Z}^d}$ the function $c \mapsto G(k, c)$ is a permutation.
3. For every $k \in S^m$ the function $s \mapsto f(k, s)$ is a permutation.

Conserved quantities

Suppose we are given a function

$$\mu : S \to \mathbb{R}$$

which assigns to each state a numeric value.

1. How do we extend μ to a function over configurations?
2. How we do this so that we can speak about conservation?

First approach: finite configurations

Suppose there is a quiescent state q.

- It is not restrictive to suppose that $\mu(q) = 0$.
  Otherwise, replace μ with $\tilde{\mu}(s) = \mu(s) - \mu(q)$.
- For every q-finite configuration c define

$$\hat{\mu}_F(c) = \sum_{\vec{n} \in \mathbb{Z}^d} \mu(c(\vec{n}))$$

Second approach: totally periodic configurations

Suppose c is totally periodic.

- Suppose that c is determined by its value on a d-hypercube D. This, as we know, is not restrictive.
- Define
$$\hat{\mu}_P(c) = \frac{1}{|D|} \sum_{\vec{n} \in D} \mu(c(\vec{n}))$$

Conserved quantities

Theorem.
Let G be a CA global function with quiescent state q
For any $\mu : S \to \mathbb{R}$ such that $\mu(q) = 0$ the following hold.

1. For every q-finite configuration c, $\hat{\mu}_F(G(c)) = \hat{\mu}_F(c)$.
2. For every totally periodic configuration c, $\hat{\mu}_P(G(c)) = \hat{\mu}_P(c)$.

In this case, $\hat{\mu}$ is a conserved quantity for the CA.

Equivalence of the two formulations

Suppose $\hat{\mu}_F(G(c)) = \hat{\mu}_F(c)$ for every $c : \mathbb{Z}^d \to S$.
Let $p : \mathbb{Z}^d \to S$ be periodic.

- Fix $k > 0$ so that p is determined by its value on a hypercube of side k.
- For $j > 0$ construct a q-finite configuration c that concides with p on a hypercube of side jk: then $\hat{\mu}_F(c) = (jk)^d \, \hat{\mu}_P(p)$
- $G(c)$ and $G(p)$ may only differ on a "hypercubic annulus" of radii $jk + 2r$ and $jk - 2r$, where r is the radius of the CA: thus,

$$\left| \hat{\mu}_F(G(c)) - (jk)^d \hat{\mu}_P(G(p)) \right| \leq 2m \cdot \left((jk + 2r)^d - (jk - 2r)^d \right)$$

- But $\hat{\mu}_F(G(c)) = \hat{\mu}_F(c) = (jk)^d \, \hat{\mu}_P(p)$. Thus,

$$|\hat{\mu}_P(p) - \hat{\mu}_P(G(p))| \leq \frac{O(j^{d-1})}{(jk)^d} \, ,$$

which is only possible if $\hat{\mu}_P(p) = \hat{\mu}_P(G(p))$.

The Hattori-Takesue criterion

Let G be the global function of a CA with quiescent state q. Let $\mu : S \to \mathbb{R}$ such that $\mu(q) = 0$.
The following are equivalent.

1. $\widehat{\mu}$ is conserved.
2. For every two finite configurations c_1, c_2 which differ in a single point,

$$\widehat{\mu}(c_1) - \widehat{\mu}(c_2) = \widehat{\mu}(G(c_1)) - \widehat{\mu}(G(c_2))$$

Rewriting the Hattori-Takesue conditions

- As $\hat{\mu}$ is clearly translation invariant, we may replace $\hat{\mu}(c_1) - \hat{\mu}(c_2)$ with $\mu(c_1(\vec{0})) - \mu(c_2(\vec{0}))$.
- If c_1 and c_2 only differ at $\vec{0}$, then $G(c_1)$ and $G(c_2)$ can only differ on those cells that have $\vec{0}$ as a neighbor. Thus,

$$\hat{\mu}(G(c_1)) - \hat{\mu}(G(c_2)) = \sum_{\vec{n} \in A} \left(\mu(G(c_1)(\vec{n})) - \mu(G(c_2)(\vec{n})) \right) ,$$

  where $A = -N = \{-\vec{n}_i \mid i = 1, \ldots, m\}$.
- It is thus possible to decide whether or not $\hat{\mu}$ is conserved, by considering all the pairs of patterns over

$$-N + N = \{\vec{n}_j - \vec{n}_i \mid i, j = 1, \ldots, m\}$$

  which only differ in $\vec{0}$.