ITT8040 Cellular Automata
Solutions to Assignment 3

Exercise 1

Let G be the global transition function of the Game of Life. The periodic configuration c such that $c(i,j) = 1$ if and only if j is a multiple of 3, is such that all cells in $G(c)$ are alive.

Exercise 2

The elementary cellular automaton rule 126 sends every three-bit string to 1, except 000 and 111, which are mapped into 0 instead. An idea is thus to find different combinations of the six strings that contain both 0 and 1, so that their image is the same. By trial and error, we find that both 00101100 and 00110100 are mapped into 111111. It is then easy to see that $\ldots 0001011000 \ldots$ and $\ldots 0001101000 \ldots$ are distinct finite configurations with the same image.

Exercise 3

We must prove that, for every choice of the integers $s, n, d, r > 0$ and every integer k large enough,

$$\left(s^{n^d} - 1\right)^{k^d} < s^{(kn-2r)^d} .$$

If $s = 1$, then the left-hand side is 0 and the right-hand side is 1 whatever k is, and there is nothing to prove. If $s > 1$, then the logarithm in base s is a strictly increasing function, and Moore's inequality is equivalent to

$$k^d \log_s \left(s^{n^d} - 1\right) < (kn - 2r)^d \ :$$

by dividing both sides by k^d, this is equivalent to

$$\log_s \left(s^{n^d} - 1\right) < \left(n - \frac{2r}{k}\right)^d ,$$

which is true for k large enough as the left-hand side is a constant strictly smaller than n^d while the right-hand side converges to n^d for $k \to \infty$.

Exercise 4

Let (S, d, N, f) be a non-surjective cellular automaton: we want to construct a configuration c that has uncountably many preimages. It is not restrictive to suppose $N = M_r$, the Moore neighborhood of radius r.

Fix $q \in S$: by the Garden-of-Eden theorem, there exist two distinct q-finite configurations e_1, e_2 with the same image h. Let k be so large that $e_1(\vec{n}) = e_2(\vec{n}) = q$ for every $\vec{n} \notin M_k$, the Moore neighborhood of radius k: if $f(q, \ldots, q) = t$, then $h(\vec{n}) = t$ for every $\vec{n} \notin M_{k+r}$. Let then $c : \mathbb{Z}^d \to S$ be the periodic configuration defined as follows:

- $c(\vec{n}) = h(\vec{n})$ for $\vec{n} \in M_{k+r}$;
- $c(\vec{n}+m\cdot(2(k+r)+1)\vec{e}_i) = c(\vec{n})$ for every $\vec{n} \in M_{k+r}$, $m \in \mathbb{Z}$, and $1 \leq i \leq d$;

Let $u_i : M_{k+r} \to S$ be the restriction of e_i to M_{k+r}: then $u_1 \neq u_2$ by construction. Let e be a configuration such that, if each coordinate of $\vec{n}$ is an integer multiple of $2(k+r)+1$, then e coincides with (a suitable translate of) either u_1 or u_2 on the hypercube of side $2(k+r)+1$ centered in $\vec{n}$: there are uncountably many such configurations, and each of them is mapped into c.