ITT8040 Cellular Automata
Solutions to Assignment 4

Exercise 1

To prove that G_F is surjective, let c be a finite configuration and let $a < b$ be integers so that all the cells $\vec{n}$ such that $c(\vec{n}) \neq (0,0)$ are inside the interval $\{a, \ldots, b\}$. Define $e : \mathbb{Z} \to \{0,1\} \times \{0,1\}$ by setting $e(\vec{n}) = c(\vec{n})$ for every $\vec{n}$ such that either $\vec{n} \notin \{a, \ldots, b\}$ or $(c(\vec{n}))_1 = 0$, and $e(\vec{n}) = ((c(\vec{n}))_0 \text{ xor } (c(\vec{n}+1))_0, 1)$ if $(c(\vec{(n)}))_1 = 1$: it is then straightforward to check that e is finite and $G_F(e) = c$.

To prove that G_P is not injective, let $c_0(\vec{n}) = (0,1)$ and $c_1(\vec{n}) = (1,1)$ for evert $\vec{n} \in \mathbb{Z}$: then c_0 and c_1 are both periodic, and $G(c_0) = G(c_1) = c_0$.

Exercise 3

It is sufficient to prove the following: for every $n \in \mathbb{Z}$ there exist $k_1, k_2 \in \mathbb{Z}$ such that $k_1 \geq n$, $k_2 \geq k_1 + m$, and for every $i \in \{0, \ldots, m-2\}$ both $c(k_1 + i) = c(k_2 + i)$ and $e(k_1 + i) = e(k_2 + i)$.

Let S be the set of states such that $c, e : \mathbb{Z} \to S$. For $j \geq 0$ consider the segments $I_{n,j} = \{n+jm, \ldots, n+(j+1)m-1\}$. Define the sequence $\eta : \mathbb{N} \to S^{2m}$ as

$$\begin{aligned} \eta(j) &= (c(n+jm), \ldots, c(n+(j+1)m-1), e(n+jm), \ldots, e(n+(j+1)m-1)) \\ &= \left(c|_{I_{n,j}}, e|_{I_{n,j}} \right) . \end{aligned}$$

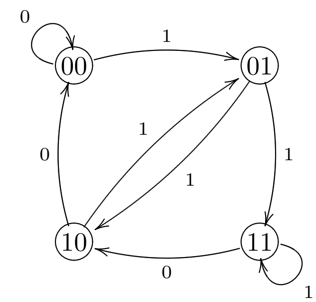
As η maps an infinite set into a finite one, there must exist a $2m$-tuple $t \in S^{2m}$ and an increasing sequence $\{j_r\}_{r \geq 0}$ such that $\eta(j_r) = t$ for every $r \geq 0$. By construction,

$$t = (c(n+j_r m), \ldots, c(n+(j_r+1)m-1), e(n+j_r m), \ldots, e(n+(j_r+1)m-1)) \ :$$

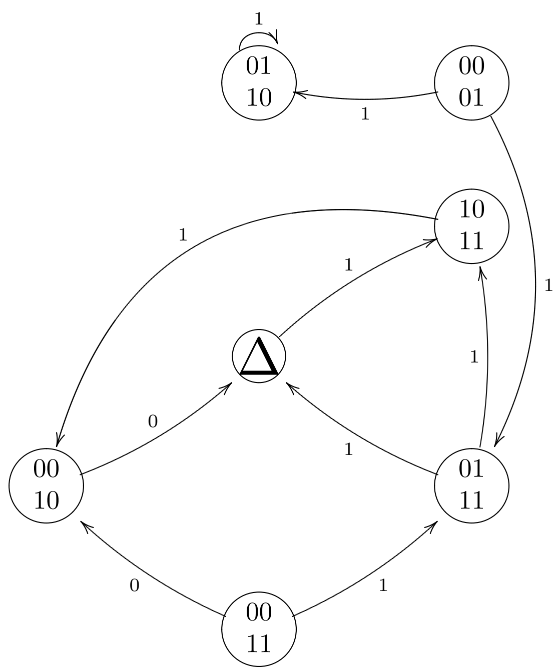
we may then set $k_1 = j_0$ and $k_2 = j_1$.

Exercise 4

The labeled de Bruijn graph of elementary CA 174 is:



Consequently, the reduced pair graph is:



By examining both graphs it can be seen that $\ldots 001000\ldots$ and $001100\ldots$ have the same image $\ldots 0110\ldots$