ITB8832 Mathematics for Computer Science
Exercise session 1: 1st September 2021

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Problems from Section 1.1

Problem 1.5.

Albert announces to his class that he plans to surprise them with a quiz sometime next week.

His students first wonder if the quiz could be on Friday of next week. They reason that it can't: if Albert didn't give the quiz before Friday, then by midnight Thursday, they would know the quiz had to be on Friday, and so the quiz wouldn't be a surprise any more.

Next the students wonder whether Albert could give the surprise quiz Thursday. They observe that if the quiz wasn't given before Thursday, it would have to be given on the Thursday, since they already know it can't be given on Friday. But having figured that out, it wouldn't be a surprise if the quiz was on Thursday either. Similarly, the students reason that the quiz can't be on Wednesday, Tuesday, or Monday. Namely, it's impossible for Albert to give a surprise quiz next week. All the students now relax, having concluded that Albert must have been bluffing. And since no one expects the quiz, that's why, when Albert gives it on Tuesday next week, it really is a surprise!

What, if anything, do you think is wrong with the students' reasoning? *Hint:* what did Albert actually say, and what did the student understand?

Problem 1.2.

What's going on here?!

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = (\sqrt{(-1)})^2 = -1\,.$$

1. Precisely identify and explain the mistake(s) in this *bogus* proof.

   Hint: What is the *context* of the equalities?

2. Prove (correctly) that if $1 = -1$, then $2 = 1$.

Problem 1.3.

Identify the bugs in the following bogus proofs:

1. **Bogus claim:** $1/8 > 1/4$.

   Bogus proof:

$$\begin{aligned} 3 &> 2 \\ 3\log_{10}(1/2) &> 2\log_{10}(1/2) \\ \log_{10}(1/2)^3 &> \log_{10}(1/2)^2 \\ (1/2)^3 &> (1/2)^2 \end{aligned}$$

   and the claim now follows by the rules for multiplying fractions.

2. *Bogus proof:* $1c = \$0.01 = (\$0.1)^2 = (10c)^2 = 100c = \$1$.

3. **Bogus claim:** If a and b are two equal real numbers, then $a = 0$.

   Bogus proof:

$$\begin{aligned} a &= b \\ a^2 &= ab \\ a^2 - b^2 &= ab - b^2 \\ (a-b)(a+b) &= (a-b)b \\ a + b &= b \\ a &= 0 \end{aligned}$$

Problem 1.4

The *arithmetic-geometric inequality* states that the arithmetic mean of two nonnegative numbers is an upper bound to their geometric mean, that is:

$$\frac{a+b}{2} \geq \sqrt{ab} \ \ \forall a, b \geq 0 \, .$$

However, there is something questionable about the following proof of this fact. What is the objection, and how would you fix it?

Bogus proof:

$$\begin{aligned} \frac{a+b}{2} &\stackrel{?}{\geq} \sqrt{ab} && \text{so} \\ a+b &\stackrel{?}{\geq} 2\sqrt{ab} && \text{so} \\ a^2+2ab+b^2 &\stackrel{?}{\geq} 4ab && \text{so} \\ a^2-2ab+b^2 &\stackrel{?}{\geq} 0 && \text{so} \\ (a-b)^2 &\stackrel{?}{\geq} 0 && \text{which we know is true}\, . \end{aligned}$$

The last statement is true because $a - b$ is a real number, and the square of a real number is never negative. This proves the claim. □

Hint: What is the conclusion that we must reach, and what are the premises?

Problems for Section 1.5

Problem 1.6.

Show that $\log_7 n$ is either an integer or irrational, where n is a positive integer. Use whatever familiar facts about integers and primes you need, but explicitly state such facts. *Hint:* Use this exercise also to review your notions from high school.

Problems for Section 1.7

Problem 1.7.

Prove by cases that

$$\max(r, s) + \min(r, s) = r + s$$

for all real numbers r, s.

Problem 1.9.

Prove by cases that $|r + s| \leq |r| + |s|$ for any two real numbers r, s.

Problem 1.10(b) (tweaked).

Suppose that

$$w^2 + x^2 + y^2 = z^2$$

where w, x, y, z are nonnegative integers. Let P the assertion that z is odd, and let R be the assertion that exactly one of w, x, and y is odd. Prove by cases that:

$$P \textbf{ iff } R.$$

Hint: An odd number equals $2m+1$ for some integer m, so its square equals $4(m^2+m)+1$.

Solutions to the problems

Problem 1.5.

The students are trying to prove that the statement "Albert will surprise us with a quiz next week" is false.

First and foremost, this is not what Albert said! Albert said that he *planned* to surprise them: not that he would do so. The students are trying to solve a problem which is a wrong version of the actual one.

But most important, the statement "Albert will surprise us with a quiz next week" *is not a proposition:* it relates to a future event, which may or may not happen, so it does not have a definite truth value.

Albert's statement, on the other hand, *is* a proposition: it says that Albert plans to do something, and that is true!

Problem 1.2.

Let's see:

1. The first two equalities are correct.

   The third equality is *seriously* wrong! First of all, it *changes the context:* negative real numbers do not have a real square root, so we are moving from real numbers to complex numbers. This could still be acceptable, if not that *in the complex field, the law of square roots does not hold:* in general, $\sqrt{ab}$ is different from $\sqrt{a}\sqrt{b}$, because every nonzero complex number has two square roots, not one. For example, if we choose the value i for the first square root of -1 and $-i$ for the second one, then their product is 1, not -1.

   The fourth equality is also wrong, because we are not sure if the square root of -1 which we chose for the first factor in the left-hand side is the same as the one chosen for the second factor.

   The last equality is correct: $\sqrt{-1}$, however we choose it, is one of the two solutions of the equation $x^2 = -1$.

2. This is one of many possible proofs. Suppose $1 = -1$. By dividing both sides by 2, $1/2 = -1/2$. By adding $3/2$, to both sides, $2 = 1$.

Problem 1.3.

(a) The error is in the second inequality. If the base is larger than 1 and the argument is smaller than 1, then the logarithm is negative: hence, the multiplication factor $\log_{10}(1/2)$ is negative. But multiplying by a negative quantity reverses the sign of the inequality, which has not been done.

(b) The error is in the wrong use of measure units. One dollar is the square of *one square root of a dollar*; similarly, one cent is the square of one square root of a cent. In turn, if one dollar corresponds to a hundred cents, then one square root of dollar corresponds to ten square root of cents, and 0.1 square root of dollars correspond to one square root of cent. The correct chain of equalities is thus:

$$1c = \$0.01 = (\sqrt{\$}\,0.1)^2 = (1\sqrt{c})^2 = 1c\,.$$

(c) Everything is fine until the second last equality. If $a = b$, then $a - b = 0$, and we cannot simplify the previous equality to get it.

Problem 1.4

The argument above is not a proof of the arithmetic-geometric inequality! Written as it is, it is, at most, *a proof of* $(a-b)^2 \geq 0$ *given the arithmetic-geometric inequality.* That is: we are going in the *wrong direction.*

To solve the issue, we observe that each inequality is *equivalent* to the next one: however given a and b, either they are both verified, or neither is. A key point here is that a and b are nonnegative reals: in particular, the critical passage from $a^2 + 2ab + b^2 \geq 4ab$ to $a + b \geq \sqrt{ab}$ is valid. We can thus replace "so" with "which is equivalent to" in the argument above, and get a proof of the arithmetic-geometric inequality.

Problem 1.6.

For what we know now, $\log_7 n$ could be either an integer, or a noninteger rational, or an irrational. The thesis is then equivalent to saying that the second case never happens: that is, if $\log_7 n$ is rational, then it is integer.

If $n = 1$ then $\log_7 1 = 0$ is integer, so we may suppose $n \geq 2$. Suppose that we can write $\log_7 n = a/b$ with a and b integers; as $n \geq 2$, we may suppose a and b both positive. By definition of logarithm, $n = 7^{a/b}$; by taking the bth powers, $n^b = 7^a$.

Now, as $a \geq 1$, the right-hand side is divisible by 7. As 7 is prime, the left-hand side must also be divisible by 7, and so must be n, because a prime number divides a product if and only if it divides one of the factors. Write $n = 7^k \cdot m$ with $k \geq 1$ and m not divisible by 7. Dividing both sides by 7^a, we obtain:

$$7^{kb-a} \cdot m^b = 1\,.$$

But the only way such equality can hold with k, a, b, and m all positive integers, is that $kb = a$ and $m = 1$. Then

$$\log_7 n = \frac{a}{b} = \frac{kb}{b} = k$$

is an integer, as we wanted to prove.

Problem 1.7.

Exactly three cases are possible: $r < s$, $r = s$, or $r > s$. Let us consider them one by one:

$r < s$ In this case, $\max(r, s) = s$ and $\min(r, s) = r$, so the equality becomes $s + r = r + s$, which is true by the commutative property.

$r = s$ In this case, the maximum and minimum of r and s are both equal to the common value t of r and s, so the equality becomes $2t = 2t$, which is trivially true.

$r > s$ In this case, $\max(r, s) = r$ and $\min(r, s) = s$, so the equality becomes $r + s = r + s$, which is trivially true.

Problem 1.9.

Recall the definition: if r is a real number, then $|r|$ equals r if $r \geq 0$, and $-r$ if $r < 0$. Equivalently: $|r| = \max(r, -r)$.

For two real numbers r and s, four cases are possible:

1. $r \geq 0$ and $s \geq 0$;
2. $r < 0$ and $s < 0$;
3. $r \geq 0$ and $s < 0$;

4. $r < 0$ and $s \geq 0$.

Correspondingly, we have four proofs:

1. If $r \geq 0$ and $s \geq 0$, then $r + s \geq 0$ too: hence,
$$|r + s| = r + s = |r| + |s| \ .$$
2. If $r < 0$ and $s < 0$, then $r + s < 0$ too: hence,
$$|r + s| = -(r + s) = -r - s = |r| + |s| \ .$$
3. If $r \geq 0$ and $s < 0$, then on the one hand,
$$r + s < r = |r| \leq |r| + |s| \ ,$$
and on the other hand,
$$-(r + s) = -r + |s| \leq |r| + |s| \ .$$
So, whether $|r + s|$ equals $r + s$ or $-(r + s)$, it is surely not larger than $|r| + |s|$.
4. If $r < 0$ and $s \geq 0$, then on the one hand,
$$r + s < s = |s| \leq |r| + |s| \ ,$$
and on the other hand,
$$-(r + s) = |r| - s \leq |r| + |s| \ .$$
So, whether $|r + s|$ equals $r + s$ or $-(r + s)$, it is surely not larger than $|r| + |s|$.

Problem 1.10(b) (tweaked).

We consider four cases according to whether w, x, and y are all even, one odd and two even, two odd and one even, or all odd.

1. If w, x, and y are all even, then so are their squares, so z^2 is even too, which is only possible if z is even in the first place. So, in this case, P and R are both false.

2. To fix ideas, suppose that w is even and x and y are odd. (The subcases where x is the odd one and where y is the odd one are treated similarly.) Then the left-hand side is the sum of one odd number and two even numbers, so it is odd; then the right-hand side is also odd, so z must be odd too. So, in this case, P and Q are both false.

3. The case where w, x and y are two odd and one even is impossible under the premise that $w^2 + x^2 + y^2 = z^2$. To see this, we exploit the suggestion and observe that the remainder of the division of a perfect square not by 2, but by 4, is either 0 or 1. In fact, let n be an integer: if $n = 2m$ is even, then the remainder of $n^2 = 4m^2$ in the division by 4 is 0; if $n = 2m + 1$ is odd, then the remainder of $n^2 = 4m^2 + 4m + 1$ in the division by 4 is 1. If w, x and y were two odd and one even, then the remainder of the division of the left-hand side by 4 would be 2, which is impossible is the sum is a perfect square.

4. For the same reason, the case where w, x and y are all odd is impossible under the premise that $w^2 + x^2 + y^2 = z^2$: if they were, then the left-hand side would be 3, while the right-hand side could only be 0 or 1.

We conclude that, in every possible case, P and R are either both true or both false; that is, P **iff** R is true.