Mathematics for Computer Science
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Problems for Section 1.8

Problem 1.17.

Prove that $\log_4 6$ is irrational.

Problem 1.19.

Prove by contradiction that $\sqrt{3}+\sqrt{2}$ is irrational. *Hint:* Expand the product $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})$.

Problems from Section 2.2

Problem 2.2 (with some small changes).

The *Fibonacci numbers* $F(0), F(1), F(2), \ldots$ are defined as follows:

$$F(n) = \begin{cases} 0 & \text{if } n = 0\,, \\ 1 & \text{if } n = 1\,, \\ F(n-1) + F(n-2) & \text{if } n > 1\,, \end{cases}$$

Exactly which sentence(s) in the following bogus proof contain logical errors? Explain.

Theorem (Bogus theorem)**.** *Every Fibonacci number is even.*

Bogus proof. Let all the variables n, m, k mentioned below be nonnegative integer valued.

1. Let $EF(n)$ mean that $F(n)$ is even.

2. Let C be the set of counterexamples to the assertion that $EF(n)$ holds for all $n \in \mathbb{N}$, namely,

$$C ::= \{n \in \mathbb{N} \mid \textbf{not}(EF(n))\}.$$

3. Assume C is nonempty. By WOP, it has a minimum m.
4. Then $m > 0$, since $F(0) = 0$ is an even number.
5. Since m is a minimum counterexample, $F(k)$ is even for all $k < m$.
6. In particular, $F(m-1)$ and $F(m-2)$ are both even.
7. But $F(m) = F(m-1) + F(m-2)$, and the right-hand side is even.
8. That is, $EF(m)$ is true, and m is not a true counterexample.
9. Then C is empty, and $F(n)$ is even for all $n \in \mathbb{N}$.

□

Problem 2.4.

Use the *Well Ordering Principle* to prove that

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{1}$$

for all nonnegative integers n.

Problems for Section 2.4

Problem 2.5

Use the Well Ordering Principle to prove that there is no solution over the positive integers to the equation:

$$4a^3 + 2b^3 = c^3.$$

Hint: Consider the smallest c such that a and b exist (if they exist at all) for which the equality above is satisfied.

Problem 2.19.

Let $\mathbb{F} = \{n/(n+1) \mid n \in \mathbb{N}\}$ and let:

$$X ::= \mathbb{N} + \mathbb{F} = \{n + f \mid n \in \mathbb{N}, f \in \mathbb{F}\},$$

where $\mathbb{F}$ is the set from Lecture 2. Prove that X is well ordered. *Hint:* You can use a trick similar to the one that worked for Problem 2.5.

Problem 2.20.

Indicate which of the following sets of numbers have a minimum element and which are well ordered. For those that are not well ordered, give an example of a subset with no minimum element.

1. The integers $\geq -\sqrt{2}$.
2. The rational numbers $\geq \sqrt{2}$. *Hint:* Between any two different real numbers there is always a rational number which is larger than the smallest of the two and smaller than the largest.
3. The set of rationals of the form $1/n$ where n is a positive integer.
4. The set G of rationals of the form m/n where $m, n > 0$ and $n \leq g$, where g is a *googol* 10^{100}. *Hint:* Consider the product M of all the positive integer from 1 to g included.

Problems from Section 3.1

Problem 3.2.

Your class has a textbook and a final exam. Let P, Q, and R be the following propositions:

- P ::= "You get an A on the final exam."
- Q ::= "You do every exercise in the book."
- R ::= "You get an A in the class."

Translate following assertions into propositional formulas using P, Q, R, and the propositional connectives **and** , **not**(), **implies** .

(a) You get an A in the class, but you do not do every exercise in the book.

(b) You get an A on the final exam, you do every exercise in the book, and you get an A in the class.

(c) To get an A in the class, it is necessary for you to get an A on the final.

(d) You get an A on the final, but you don't do every exercise in this book; nevertheless, you get an A in this class.

Solutions

Problem 1.17.

By contradiction, assume $\log_4 6 = m/n$ for suitable positive in- tegers m, n. As we saw during the lecture, we may suppose also that m and n are positive and relatively prime. Then $6^n = 4^{n \log_4 6} = 4^m$, which is impossible, because 6^n is divisible by 3, and 4^m is not.

Important note: As 4 is not prime, we *cannot* conclude that, since 6 is not an integer power of 4, then $\log_4 6$ is irrational. For example, 8 is not an integer power of 4, but $\log_4 8 = \frac{\log_2 8}{\log_2 4} = \frac{3}{2}$ is rational.

Problem 1.19.

We follow the hint and perform the multiplication:

$$(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2}) = 3 - 2 = 1\,.$$

This means that $\sqrt{3} - \sqrt{2}$ is the multiplicative inverse of $\sqrt{3} + \sqrt{2}$. By contradiction, assume $\sqrt{3} + \sqrt{2} = m/n$ is rational. Then $\sqrt{3} - \sqrt{2} = n/m$ is rational too, and so is their difference $2\sqrt{2}$. But then, so is $\sqrt{2}$: contradiction.

If, instead of the difference $2\sqrt{2}$, we consider the sum $2\sqrt{3}$, we reach a similar contradiction. Indeed, an argument similar to our proof of the irrationality of $\sqrt{2}$ leads us to the conclusion that $\sqrt{3}$ is rational.

Problem 2.2 (with some small changes).

The problem is with point 6. Until now, we only know that m is positive: it could well be 1. (It is so indeed, but that's not the point.) But if $m = 1$, then $m - 2 = -1$ is not a natural number; and we have only defined the Fibonacci numbers as a function on the naturals, not on all integers! For what we know, $F(m - 2)$ might not exist.[1]

Problem 2.4.

First, a note on notation. Let a be an integer, and for each integer $k \geq a$ such that $a \leq k \leq b$ let x_k be a complex number. Then for $n \geq a$ integer the

[1] As a curiosity: it is possible to define the Fibonacci numbers on negative integers, and it turns out that it must be $F(-1) = 1$. More in general, if n is a positive integer, then $F(-n) = (-1)^{n-1} F(n)$.

sum of the numbers x_k for k from a to n is defined as follows:

$$\sum_{k=a}^{a} x_k = x_a \,; \quad \sum_{k=a}^{n} x_k = \left(\sum_{k=a}^{n-1} x_k \right) + x_n \ \text{ for every } n > a \,.$$

This is an example of a *recursive definition*, where the current value is constructed from the previous ones. We will see more of these in later chapters.

Let C be the set of counterexamples to (1), namely,

$$C ::= \left\{ n \in \mathbb{N} \mid \sum_{k=0}^{n} k^2 \neq \frac{n(n+1)(2n+1)}{6} \right\} .$$

If C is nonempty, then it has a minimum element m: such m must be positive, because for $n = 0$ both sides of (1) are zero. Since m is the minumum of C, $m - 1$, which is still a natural number as m is positive, *does* satisfy (1): we then have

$$\sum_{k=0}^{m-1} k^2 = \frac{(m-1)m(2(m-1)+1)}{6} .$$

But then,

$$\begin{aligned}
\sum_{k=0}^{m} k^2 &= \sum_{k=0}^{m-1} k^2 + m^2 \\
&= \frac{(m-1)m(2(m-1)+1)}{6} + m^2 \\
&= \frac{(m^2-m)(2m-1)+6m^2}{6} \\
&= \frac{2m^3-3m^2+m+6m^2}{6} \\
&= \frac{2m^3+3m^2+m}{6} \\
&= \frac{m(2m^2+3m+1)}{6} \\
&= \frac{m(m+1)(2m+1)}{6} :
\end{aligned}$$

that is m *does* satisfy (1) after all. The contradiction stems from our hypothesis that C be nonempty: hence, c is empty, and (1) holds for every nonnegative integer m.

Problem 2.5

Let c_0 be the smallest positive integer such that positive integers a_0 and b_0 exist such that $4a_0^3 + 2b_0^3 = c_0^3$. We observe that $c_0 > 1$, because the left-hand side must be even: indeed, c_0 itself must be even, so it must be $c_0 = 2c_1$ for some positive integer c_1. We then have:

$$4a_0^3 + 2b_0^3 = 8c_1^3 \,,$$

which, dividing by 2, yields:

$$2a_0^3 + b_0^3 = 4c_1^3 \,.$$

Now, the right-hand side is even, so both summands on the left-hand side must be even: this means that b_0 must be even, so we write $b_0 = 2b_1$ for a suitable positive integer h. Again, we get, first, $2a_0^3 + 8b_1^3 = 4c_1^3$, then, dividing by 2,

$$a_0^3 + 4b_1^3 = 2c_1^3 \,.$$

This time, with the same logic, $a_0 = 2a_1$ for a suitable positive integer a_1: substituting and replacing, we find... guess what?,

$$4a_1^3 + 2b_1^3 = c_1^3 \,,$$

which is a solution over the positive integers with $c_1 < c_0$. We have thus discovered that the smallest counterexample c_0 was not the smallest: then there was no c_0 in the first place, and the equation does not have a solution on the positive integers.

Problem 2.19.

Let S be a nonempty set of $\mathbb{N} + \mathbb{F}$. Then $n + f \in S$ for *some* $n \in \mathbb{N}$ and $f \in \mathbb{F}$: let n_S be the *minimum* nonnegative integer such that $n_S + f \in S$ for some $f \in \mathbb{F}$. Since $\mathbb{F}$ is also well ordered, there exists a *minimum* $f_S \in \mathbb{F}$ such that $n_S + f_S \in S$. We will show that $n_S + f_S$ is the minimum of S.

Let $s = n + f \in S$. By definition of n_S, is it $n_S \leq n$. If $n_S < n$, then $n_S + f_S < s$ too, because $f_S < 1$. If $n = n_S$ instead, then by definition of f_S it is $f_S \leq f$, so $n_S + f_S \leq n_S + f = s$.

Problem 2.20.

Let's see:

1. An integer m is greater or equal to $-\sqrt{2}$ if and only if it is greater or equal to -1. We know from the lecture that $\{m \in \mathbb{Z} \mid m \geq -1\}$ is well ordered.

2. This set is not well ordered. To see why, put $x_0 = 2$, $x_1 = 1.5$, $x_2 = 1.42$, $x_3 = 1.415$, $x_4 = 1.4143$, and so on: in general, let x_n be made of the decimal writing of $\sqrt{2}$ up to the nth decimal digit rounded up. Then $x_n > \sqrt{2}$ for every $n \in \mathbb{N}$, but the set $S = \{x_n \mid n \in \mathbb{N}\}$ does not have a minimum, because for every element there is a strictly smaller element.

   Alternatively: for every $n \geq 0$ let a_n be the truncation to the nth decimal digit of the decimal expansion of $\sqrt{2}$, so that $a_0 = 1$, $a_1 = 1.4$, $a_2 = 1.41$, $a_3 = 1.414$, and so on. Let $x_n = 3 - a_n$: then $x_n > \sqrt{2}$ for every $n \geq 0$, because

   $$3 - a_n > 3 - \sqrt{2} > 3 - \frac{3}{2} = \frac{3}{2} > \sqrt{2}\,.$$

   But the set $\{x_n \mid n \geq 0\}$ does not have a minimum, because if $m < n$ then $x_m > x_n$.

3. This set is not well ordered: no point $x = 1/n$ can be the minimum, because $1/(n+1) < 1/n$ if n is a positive integer.

4. The set G is well ordered! To see this, let:

   $$a = 1 \cdot 2 \cdots (g-1) \cdot g = g!$$

   be the *factorial* of g, that is, the product of all positive integers from 1 to g included. Then, since $n \leq g$ when $x = m/n \in G$, for every such x the number ax is a positive integer; also, $x \leq y$ if and only if $ax \leq ay$. So, however given a nonempty subset S of G, the set $T = \{ax \mid x \in S\}$ is a nonempty subset of positive integers: if m is the minimum of T, then m/a is the minimum of S.

Problem 3.2.

(a) In this case, R is verified, Q is not, and P is irrelevant: the assertion translates as R **and not**(Q).

(b) Here P, Q, and R are all verified, so this assertion translates as P **and** Q **and** R. Recall that **and** is associative, so (P **and** Q) **and** R is equivalent to P **and** (Q **and** R).

(c) Here we have a clear implication, and a causal one too! What the assertion says, is that if you get an A in the class, it means that you had gotten an A in the final: the translation in mathematical language is then R **implies** P.

(d) In this case, P is true, Q is false, and R is true: the assertion translates as P **and not**(Q) **and** R.

At the end of the exercise, observe how, in mathematical language, "but" and "nevertheless" mean the same as "and". The differences in tone of the three words are lost in translation.