Mathematics for Computer Science
Exercise session 4, 22 September 2021

Silvio Capobianco

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Problems from Section 4.1

Problem 4.3.

(a) Verify that the propositional formula $(P \textbf{ and } \overline{Q}) \textbf{ or } (P \textbf{ and } Q)$ is equivalent to P.

(b) Prove that

$$A = (A - B) \cup (A \cap B)$$

for all sets A, B, by showing

$$x \in A \textbf{ iff } x \in (A - B) \cup (A \cap B)$$

for all elements x using the equivalence of part (a) in a chain of **iff** 's.

Problem 4.6.

Let A and B be sets.

(a) Prove that

$$\text{pow}(A \cap B) = \text{pow}(A) \cap \text{pow}(B) .$$

(b) Prove that

$$\text{pow}(A) \cup \text{pow}(B) \subseteq \text{pow}(A \cup B) ,$$

with equality holding iff one of A or B is a subset of the other.

Problem 4.7 (introduction).

The game of *Subset Take-Away* is played between two players with the following rules:

1. The initial position is a finite nonempty set.
2. Taking turns, the players take away subsets of the initial set.
3. It is not permitted to take away the entire initial set as the first move.
4. Once a subset has been taken away, no subset which contains it can be taken away anymore.
5. A player who cannot take away a nonempty subset on his or her turn, loses the game.

Note that rule 4 implies that any subset of the initial set can be taken out at most once.

1. Explain why a game which starts from a set with one or two elements is a win for the second player.
2. Expand your argument to prove that a game starting with a set of three elements is again a win for the second player.

Note that the book claims that it is is unsolved whether the second player has a winning strategy for any initial position. However, this has been disproved in 2017: if the initial set has 7 elements, then the first player has a winning strategy. See also `https://www.win.tue.nl/~aeb/games/chomp.html`

Problems for Section 4.2

Problem 4.14.

Prove that for any sets A, B, C and D, if the Cartesian products $A \times B$ and $C \times D$ are disjoint, then either A and C are disjoint or B and D are disjoint.

Problem 4.15(a).

Give a simple example where the following result fails, and briefly explain why:

False Theorem. *For sets A, B, C and D, let*

$$\begin{aligned} L &::= (A \cup B) \times (C \cup D), \\ R &::= (A \times C) \cup (B \times D). \end{aligned}$$

Then $L = R$.

Problem 4.16.

The *inverse* R^{-1} of a binary relation R from A to B is the relation from B to A defined by:

$$bR^{-1}A \textbf{ iff } aRb$$

In other words, you get the diagram for R^{-1} from R by "reversing the arrows" in the diagram describing R. Now many of the relational properties of R correspond to different properties of R^{-1} . For example, R is *total* iff R^{-1} is a *surjection*.

Fill in the remaining entries in this table:

R is	iff R^{-1} is
total	a surjection
a function	
a surjection	
an injection	
a bijection	

Hint: Explain what's going on in terms of "arrows" from A to B in the diagram for R.

Problem 4.29(a)-(e).

The language of sets and relations may seem remote from the practical world of programming, but in fact there is a close connection to *relational databases*, a very popular software application building block implemented by such software packages as MySQL. This problem explores the connection by considering how to manipulate and analyze a large data set using operators over sets and relations. Systems like MySQL are able to execute very similar high-level instructions efficiently on standard computer hardware, which helps programmers focus on high-level design.

Consider a basic Web search engine, which stores information on Web pages and processes queries to find pages satisfying conditions provided by users. At a high level, we can formalize the key information as:

- A set P of *pages* that the search engine knows about
- A binary relation L (for *link*) over pages, defined such that $p_1 L p_2$ if and only if p_1 links to p_2
- A set E of *endorsers*, people who have recorded their opinions about which pages are high-quality
- A binary relation R (for *recommends*) between endorsers and pages, such that eRp iff person e has recommended page p
- A set W of *words* that may appear on pages
- A binary relation M (for *mentions*) between pages and words, where pMw iff word w appears on page p

Each part of this problem describes an intuitive, informal query over the data, and your job is to produce a single expression using the standard set and relation operators, such that the expression can be interpreted as answering the query correctly, for any data set. Your answers should use only the set and relation symbols given above, in addition to terms standing for constant elements of E or W, plus the following operators introduced in the text:

- set union $\cup$.
- set intersection $\cap$.
- set difference $-$.
- relational image—for example, $R(A)$ for some set A, or $R(a)$ for some specific element a.
- relational inverse $^{-1}$.

- ...and one extra: *relational composition* which generalizes composition of functions

$$a\,(R \circ S)\,c ::= \exists b \in B \,.\, a\,S\,b \textbf{ and } b\,R\,c\,.$$

In other words, a is related to c in $R \circ S$ if starting at a you can follow an S arrow to the start of an R arrow and then follow the R arrow to get c.[1]

Here is one example to get you started:

- **Search description:** The set of pages contains the word "logic".
- **Solution expression:** M^{-1}("logic")

Find similar solutions for each of the following searches:

(a) The set of pages containing the word "logic" but not the word "predicate".

(b) The set of pages containing the word "set" that have been recommended by "Meyer".

(c) The set of endorsers who have recommended pages containing the word "algebra".

(d) The relation that relates endorser e and word w iff e has recommended a page containing w.

(e) The set of pages that have at least one incoming or outgoing link.

Problems for Section 4.5

Problem 4.39

Let $A = \{a_0, a_1, \ldots, a_{n-1}\}$ be a set of size n, and $B = \{b_0, b_1, \ldots, b_{m-1}\}$ a set of size m. Prove that $|A \times B| = mn$ by defining a simple bijection from $A \times B$ to the nonnegative integers from 0 to $mn - 1$.

[1] Note the reversal of R and S in the definition: this is to make relational composition work like function composition. For functions, $f \circ g$ means you apply g first. That is, if we let h be $f \circ g$, then $h(x) = f(g(x))$.

Solutions

Problem 4.3.

(a) By using distributivity:

$$\begin{aligned}(P \textbf{ and } \overline{Q}) \textbf{ or } (P \textbf{ and } Q) \quad & \textbf{iff} \quad P \textbf{ and } (\overline{Q} \textbf{ or } Q) \\ & \textbf{iff} \quad P \textbf{ and } \mathbf{T} \\ & \textbf{iff} \quad P\,.\end{aligned}$$

(b) Let $P ::= x \in A$ and $Q ::= x \in B$: then,

$$\begin{aligned} x \in A \quad & \textbf{iff} \quad (x \in A \textbf{ and not}(x \in B)) \textbf{ or } (x \in A \textbf{ and } x \in B) \\ & \textbf{iff} \quad (x \in A - B) \textbf{ or } (x \in A \cap B) \\ & \textbf{iff} \quad x \in (A - B) \cup (A \cap B)\,. \end{aligned}$$

Problem 4.6.

(a) Let S be a set. We must prove:

$$S \subseteq A \cap B \textbf{ iff } S \subseteq A \textbf{ and } S \subseteq B \tag{1}$$

We can better do this[2] by proving the equivalence as a double implication:

$$(S \subseteq A \cap B \longrightarrow S \subseteq A \wedge S \subseteq B) \wedge (S \subseteq A \wedge S \subseteq B \longrightarrow S \subseteq A \cap B) \tag{2}$$

Suppose the left-hand side of (1) holds. Let x be an arbitrary element: if $x \in S$, then $x \in A \cap B$, so both $x \in A$ and $x \in B$ by definition of intersection. We have thus proved that, if $S \subseteq A \cap B$, then $S \subseteq A$ **and** $S \subseteq B$: that is, $\text{pow}(A \cap B) \subseteq \text{pow}(A) \cap \text{pow}(B)$.

Suppose now that the right-hand side of (1) holds. Recall that such intersection is never empty, because the empty set is a subset of every set, thus an element of every power set. Let $S \subseteq A$ and $S \subseteq B$: if S is empty, then $S \subseteq A \cap B$ for sure; if S is not empty, then every element of S belongs to both A and B, thus to $A \cap B$, and this shows $S \subseteq A \cap B$. We have thus proved that, if $S \subseteq A$ **and** $S \subseteq B$, then $S \subseteq A \cap B$: that is, $\text{pow}(A) \cap \text{pow}(B) \subseteq \text{pow}(A \cap B)$. Double inclusion means equality.

[2]The classroom discussion depends on a passage which is not immediate to justify.

(b) Let S be a set. We must prove:

$$S \subseteq A \textbf{ or } S \subseteq B \textbf{ implies } S \subseteq A \cup B \tag{3}$$

But this is easy to see: if $S \subseteq A$, then for every $x \in S$ it is also $x \in A$, thus $x \in A \cup B$ as well, and as x is arbitrary, $S \subseteq A \cup B$. Similarly, if $S \subseteq B$, then $S \subseteq A \cup B$.

Now, if for some $x \in A$ it is $x \notin B$, then any subset of $A \cup B$ which has x as an element cannot be a subset of B. It might still be, however, that every element of B is also an element of A: in this case, $A \cup B = A$ and $S \subseteq B$ **implies** $S \subseteq A$, so:

$$\text{pow}(A) \cup \text{pow}(B) = \text{pow}(A) = \text{pow}(A \cup B)\,.$$

That is: if $B \subseteq A$, then the inclusion at (b) is an equality. The same holds, with the roles of A and B swapped, if $A \subseteq B$. However, if neither $A \subseteq B$ nor $B \subseteq A$, then there exist $x \in A$ and $y \in A$ such that $x \notin B$ and $y \notin A$: in this case, $\{x, y\}$ is a subset of $A \cup B$, but not a subset of A nor of B, and the inclusion is strict.

Problem 4.7 (introduction).

1. If the initial set has only one element, the first player has no legal moves from the start. If the initial set has the form $\{a, b\}$, then the first move of the first player is either $\{a\}$ or $\{b\}$; the second player takes away the remaining singleton and wins the game.

2. If the initial set has three elements, say $\{a, b, c\}$, then the first player can take away either a subset with one element, or a subset with two elements.

   In the first case, let's say that the first player takes away $\{a\}$. This eliminates the moves $\{a, b\}$ and $\{a, c\}$, so any other move must be a subset of $\{b, c\}$. If the second player chooses $\{b, c\}$, they reduce the original game to a game starting from a set with two elements, for which they have a winning strategy.

   In the second case, let's say that the first player takes away $\{a, b\}$. If the second player takes away $\{c\}$, they make the moves $\{b, c\}$ and $\{a, c\}$ impossible, so any further move must be a subset of $\{a, b\}$. Again the second player has reduced the original game to a game starting from a set with two elements, for which they have a winning strategy.

Problem 4.14.

Recall that two sets are disjoint if they have no common elements. Let's prove the contrapositive: if A and C are not disjoint and B and D are not disjoint, then $A \times B$ and $C \times D$ are not disjoint. And this is easy to see: if $x \in A \cap C$ and $y \in B \cap D$, then $(x, y) \in (A \times B) \cap (C \times D)$.

Problem 4.15(a).

If A and D are empty, but B and C are not, then L is not empty and R is. There is no such thing as a pair without a first element, or without a second element.

The problem here is that the choices for the left and right component are independent in L, but not in R. In L, if we have chosen the first component from A, then we still have the option of choosing the second component from either C or D: but in R, we are forced to choose it from C.

Problem 4.16.

We preliminarily observe that $(R^{-1})^{-1} = R$, as:

$$a\,(R^{-1})^{-1}\,b \textbf{ iff } b\,R^{-1}\,a \textbf{ iff } a\,R\,b$$

Then we can immediately fill:

R is	iff R^{-1} is
total	a surjection
a function	
a surjection	total
an injection	
a bijection	

To fill the rest of the table, we observe that the relation diagram of R^{-1} is obtained from that of R by first reflecting it along a vertical line which cuts the arrows in half, then reversing the direction of each arrow. This leads to the following important observation:

R has the $\star n$ **in** property if and only if R^{-1} has the $\star n$ **out** property where $\star$ is either $\leq$, $\geq$, or $=$. As the inverse of the inverse relation is the original relation, the observation above also holds with the roles of R and R^{-1} swapped.

We can now go on:

$$\begin{aligned} R \text{ is a function} \quad & \textbf{iff} \quad R \text{ has the } \leq 1 \textbf{ out } \text{ property} \\ & \textbf{iff} \quad R^{-1} \text{ has the } \leq 1 \textbf{ in } \text{ property} \\ & \textbf{iff} \quad R^{-1} \text{ is an injection} \end{aligned}$$

To conclude, we recall that a bijection is a total function which is both injective and surjective: in this case, R^{-1} is a surjective and injective relation which is both a function and total, so it is also a bijection. And vice versa. The final table is thus:

R is	iff R^{-1} is
total	a surjection
a function	an injection
a surjection	total
an injection	a function
a bijection	a bijection

Problem 4.29(a)-(e).

Let's go through the points one by one:

(a) We want the pages which mention "logic" but do not mention "predicate". This corresponds to the difference set of $M^{-1}(\text{"logic"})$ with $M^{-1}(\text{"predicate"})$. So the set we need is:

$$A ::= M^{-1}(\text{"logic"}) - M^{-1}(\text{"predicate"})\,.$$

(b) We want the pages which not only contain the word "set", but are also recommended by Meyer. This corresponds to the intersection of $M^{-1}(\text{"set"})$ of the pages where the word "set" is mentioned with the set $R(\text{Meyer})$ of the pages which Meyer recommends. So the set we need is:

$$B ::= M^{-1}(\text{"set"}) \cap R(\text{Meyer})\,.$$

(c) We have to make two steps here: first, identify the pages which contain the word "algebra"; then, identify the people who endorse those pages.

We know that the set of the pages which contain a word w is $M^{-1}(w)$, and that the set of endorsers of a page p is $E^{-1}(p)$. Thus, to find the set of endorsers who have recommended pages containing the word "algebra" we first apply M^{-1} to "algebra", then E^{-1} to M^{-1}("algebra"). So the set we need is:

$$C ::= E^{-1} \circ M^{-1}(\text{“algebra”})\,.$$

(d) Call D the relation we want to find. We know that eDw if and only if there exists p such that pMw and eRp. That is:

$$D ::= \{(e,w) \mid \exists w\,.\,eRp \textbf{ and } pMw\} = M \circ R\,.$$

(e) A page p has an incoming link if and only if there exists a page q such that qLp, and has an outgoing link if and only if there exists a page r such that pLr. The set of the q's which satisfy qLp is $L^{-1}(p)$, and the set of the r's which satisfy pLr is $L(p)$. As at least one of these must happen, the relation we look for is:

$$S ::= L^{-1}(p) \cup L(p)\,.$$

Problem 4.39

We observe that we can order the elements of $A \times B$ into a *matrix* with n rows and m columns:

$$\begin{pmatrix} (a_0,b_0) & (a_0,b_1) & (a_0,b_2) & \dots & (a_0,b_{m-1}) \\ (a_1,b_0) & (a_1,b_1) & (a_1,b_2) & \dots & (a_1,b_{m-1}) \\ (a_2,b_0) & (a_2,b_1) & (a_2,b_2) & \dots & (a_2,b_{m-1}) \\ \vdots & & & & \vdots \\ (a_{n-1},b_0) & (a_{n-1},b_1) & (a_{n-1},b_2) & \dots & (a_{n-1},b_{m-1}) \end{pmatrix} \tag{4}$$

But we can do the same with the natural numbers smaller than mn:

$$\begin{pmatrix} 0 & 1 & 2 & \dots & m-1 \\ m & m+1 & m+2 & \dots & 2m-1 \\ 2m & 2m+1 & 2m+2 & \dots & 3m-1 \\ \vdots & & & & \vdots \\ (n-1)m & (n-1)m+1 & (n-1)m+2 & \dots & mn-1 \end{pmatrix} \tag{5}$$

Each possible pair (a_i, b_j) appears exactly once in the matrix (4). Each possible natural number smaller than mn appears exactly once in the matrix (5). Then we can obtain a bijection between $A \times B$ and $\{0, \dots, mn-1\}$ by *superimposing the matrices*. If we do so, we notice that the pair (a_i, b_j) corresponds to the number $mi + j$: this is the bijection we were looking for.