ITB8832 Mathematics for Computer Science
Autumn 2021, First test

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Exercise 1 (3 points)

Use the Well Ordering Principle to prove that for every integer $n \geq 1$

$$\sum_{k=1}^{n} (-1)^k \cdot k^2 = (-1)^n \cdot \frac{n(n+1)}{2} \tag{1}$$

Important: Any solution which does not use the Well Ordering Principle will receive zero points.

Exercise 2 (3 points)

Let F be a propositional formula depending on the propositional variables $P_1, \ldots, P_n$. If we replace each propositional variable P_i inside F with a predicate formula $Q_i(x)$ which depends on a variable x (the same for all predicates) we obtain a predicate formula in the variable x, which we call $G(x)$. For example, if:

$$F = (P_1 \textbf{ and } P_2) \textbf{ or } P_3 \,,$$

then:

$$G(x) = (Q_1(x) \textbf{ and } Q_2(x)) \textbf{ or } Q_3(x) \,.$$

Prove that if $\forall x \,.\, G(x)$ is valid, then F is valid. *Hint:* prove the contrapositive by constructing a counter-model for $\forall x \,.\, \Phi(x)$.

Exercise 3 (3 points)

Let A, B, and C be sets. Prove that:

$$(A - B) \cup (A - C) = A - (B \cap C) \tag{2}$$

Exercise 4 (6 points overall)

For each of the following questions, mark the only correct answer:

1. (1 point) Which one of the following numbers is rational, but not integer?
   (a) $\log_5 25$.
   (b) $\log_5 27$
   (c) $\log_{25} 5$.
2. (1 point) Which one of the following sets is well ordered?
   (a) $A ::= \{x \in \mathbb{Z} \mid \exists n \in \mathbb{N} \,.\, x + n^2 = 0\}$.
   (b) $B ::= \{x \in \mathbb{R} \mid \exists n \in \mathbb{N} \,.\, (n+1)x = n\}$.
   (c) $C ::= \{x \in \mathbb{R} \mid \exists n \in \mathbb{N} \,.\, nx = 1\}$.
3. Which one of the following formulas is equivalent to P **implies** (Q **implies** R)?
   (a) (P **and** Q) **implies** R.
   (b) (P **implies** Q) **implies** R.
   (c) (P **or** Q) **implies** R.
4. Which one of the following predicate formulas is valid?
   (a) $(\exists x \,.\, \forall y \,.\, P(x,y))$ **implies** $(\forall y \,.\, \exists x \,.\, P(x,y))$.
   (b) $(\forall x \,.\, \exists y \,.\, P(x,y))$ **implies** $(\forall y \,.\, \exists x \,.\, P(x,y))$.
   (c) $(\forall y \,.\, \exists x \,.\, P(x,y))$ **implies** $(\exists x \,.\, \forall y \,.\, P(x,y))$.
5. Which one of the following relations is a bijection?
   (a) $R : \mathbb{R} \to \mathbb{R}\,, xRy$ **iff** $y = x^2$.
   (b) $S : \mathbb{R} \to \mathbb{R}\,, xSy$ **iff** $y = x^3$.
   (c) $T : \mathbb{R} \to \mathbb{R}\,, xTy$ **iff** $y = x^{1/2}$.
6. Let A and B be finite sets and let $R : A \to B$ be a total injective relation. Which of the following is true whatever A and B are?
   (a) $|R(A)| = |B|$.
   (b) $|R(A)| < |B|$.
   (c) $|A| \leq |B|$.