Mathematics for Computer Science
Self-evaluation exercises for Lecture 2

Silvio Capobianco

Last update: 23 August 2022

Exercises on Chapter 1

Exercise 1.1 (from the classroom test of 03.10.2018)

1. Prove that $\log_{20} 50$ is irrational.
2. Let $a > 1$ and $b > 1$ be integers. Can $\log_a b$ be rational if b is not a power of a?

Exercise 1.2 (cf. Problems 1.8 and 1.23)

Use the following two arguments to prove that, if a and b are both irrational positive real numbers, then a^b can still be rational:

1. Consider $x = y = \sqrt{2}$ and consider the cases where x^y is rational and when it is irrational.
2. Prove that $b = \log_2 9$ is irrational and make a good choice for a.

What difference can you see between the two proofs?

Exercise 1.3 (cf. Problem 1.19)

An integer m is a *divisor* of an integer n if there exists an integer k such that $m \cdot k = n$. Note that, with this definition, every integer is a divisor of 0.

Let $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_d x^d$ be a polynomial of degree $d \geq 1$ with integer coefficients. The *rational root theorem* says that, if for two relatively prime integers m, n the value m/n is a root of $p(x)$ (that is, $p(m/n) = 0$) then m is a divisor of the *constant term* a_0 and n is a divisor of the *leading coefficient* a_d.

1. Prove the rational root theorem.

2. Use the rational root theorem to prove that, if the integer k is not the rth power of some other integer, then the rth root of k is irrational. *Hint:* prove the contrapositive.

   Note: you *do not* need to have solved point 1 before you solve point 2, but you *must* use it.

Exercises on Chapter 2

Exercise 2.1 (cf. Problem 2.6)

You are given a series of envelopes, respectively containing $1, 2, 4, \ldots, 2^m$ dollars. Define

> **Property** m: For any nonnegative integer less than 2^{m+1}, there is a selection of envelopes whose contents add up to *exactly that number of dollars.*

Use the Well Ordering Principle (WOP) to prove that Property m holds for all nonnegative integers m.

Hint: Consider two cases: first, when the target number of dollars is less than 2^m, and second, when the target is at least 2^m.

Exercise 2.2 (from the midterm test of 07.10.2019)

Let a be a real number, different from 1. Use the Well Ordering Principle to prove that, for every nonnegative integer n,

$$1 + a + \ldots + a^n = \frac{1 - a^{n+1}}{1 - a} \,. \tag{1}$$

Important: solutions which do not use the Well Ordering Principle will receive zero points.

Exercise 2.3 (cf. problem 2.22)

Let S be a subset of the set of real numbers. An *infinite descent* in S is an infinite sequence $\{s_n \mid n \in \mathbb{N}\}$ of elements of S such that:

$$s_n > s_{n+1} \text{ for every } n \in \mathbb{N} \,. \tag{2}$$

Prove that S is well ordered if and only if it *does not* have an infinite descent.

Exercise 2.4

You meet a man whom you know to be either a *knight* who only makes true statements, or a *knave* who only makes false statements (but you don't know which of the two). The man makes the following statement:

> "Today is not the first day on which I make this statement."

Is he a knight or a knave? *Hint:* choose a "good" subset of the set of natural numbers and use the Well Ordering Principle.