Mathematics for Computer Science
Self-evaluation exercises for Chapter 4

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Exercise 4.1 (from the classroom test of 3 October 2018)

Let A and B be nonempty sets and n a positive integer. Prove that the relation $R : A^n \times B^n \to (A \times B)^n$ defined by:

$$(\alpha, \beta)\, R\, \gamma \quad \textbf{iff} \quad c_i = (a_i, b_i) \text{ for every } i \in \mathbb{N}, 1 \leq i \leq n\,,$$

where $\alpha = (a_1, \ldots, a_n) \in A^n$, $\beta = (b_1, \ldots, b_n) \in B^n$, and $\gamma = (c_1, \ldots, c_n) \in (A \times B)^n$, is a bijection.

Exercise 4.2 (cf. Problem 4.18)

For each of the following real-valued functions on the real numbers, indicate whether it is a bijection, a surjection but not a bijection, an injection but not a bijection, or neither an injection nor a surjection.

(a) $x \mapsto x + 2$

(b) $x \mapsto 2x$

(c) $x \mapsto x^2$

(d) $x \mapsto x^3$

(e) $x \to \sin x$

(f) $x \to x \sin x$

(g) $x \to e^x$

Exercise 4.3 (from the midterm test of 7 October 2020)

Use the Well Ordering Principle to prove the following: if n is a positive integer and $A, B_1, B_2, \ldots, B_n$, are arbitrary sets, then

$$A \cap (B_1 \cup B_2 \cup \ldots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_n) .$$

Hint: start with proving that, if m is the minimum counterexample, then $m \geq 3$.

Exercise 4.4 (cf. Problem 4.7)

Recall that the Subset Take-Away game has the following rules:

1. The initial position is a finite nonempty set.
2. Taking turns, the players take away subsets of the initial set.
3. It is not permitted to take away the entire initial set as the first move.
4. Once a subset has been taken away, no subset which contains it can be taken away anymore.

   In particular, every subset can be taken away at most once.
5. A player who cannot take away a nonempty subset on his or her turn, loses the game.

We have seen in classroom that, if the initial position has either one, two, or three elements, then the second player has a winning strategy. Prove that the second player still has a winning strategy if the initial position has four elements. *Hint:* determine three cases according to what kind of initial move the first player does; two can be easily reduced to a previously solved game, while the third requires more attention.

Exercise 4.5 (from the midterm test of 07.10.2019)

1. Show two sets A and B such that **not**$((A \times B)$ inj $(A \cup B))$.
2. Show two sets A and B such that **not**$((A \cup B)$ inj $(A \times B))$.

Hint: choose finite sets and work with cardinalities.

Exercise 4.6 (cf. Problem 4.29(f)-(h))

The basic Web search engine which we discussed in exercise session 4 can be described in terms of:

- A set P of *pages* that the search engine knows about
- A binary relation L (for *link*) over pages, defined such that $p_1 L p_2$ if and only if p_1 links to p_2
- A set E of *endorsers*, people who have recorded their opinions about which pages are high-quality
- A binary relation R (for *recommends*) between endorsers and pages, such that eRp iff person e has recommended page p
- A set W of *words* that may appear on pages
- A binary relation M (for *mentions*) between pages and words, where pMw iff word w appears on page p

Use this specification to express the following relations:

1. The relation that relates word w and page p iff w appears on a page that links to p.
2. The relation that relates word w and endorser e iff w appears on a page that links to a page that e recommends.
3. The relation that relates pages p_1 and p_2 iff p_2 can be reached from p_1 by following a sequence of exactly 3 links.

Exercise 4.7 (cf. Problem 4.38)

Assume $f : A \to B$ is total function, and A is finite. Replace the $\star$ with one of $\leq$, $=$, $\geq$ to produce the *strongest* correct version of the following statements:

(a) $|f(A)| \star |B|$.

(b) If f is a surjection, then $|A| \star |B|$.

(c) If f is a surjection, then $|f(A)| \star |B|$.

(d) If f is an injection, then $|f(A)| \star |A|$.

(e) If f is a bijection, then $|A| \star |B|$.