ITT9132 Concrete Mathematics
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A note on the repertoire method

Suppose that we have a recursion scheme of the form:

$$\begin{aligned} g(0) &= \alpha \,, \\ g(n+1) &= \Phi(g(n)) + \Psi(n; \beta, \gamma, \ldots) \quad \text{for } n \geq 0 \,. \end{aligned} \tag{1}$$

Suppose now that:

1. Φ is linear *in* g, *i.e.*, if $g(n) = \lambda_1 g_1(n) + \lambda_2 g_2(n)$ then $\Phi(g(n)) = \lambda_1 \Phi(g_1(n)) + \lambda_2 \Phi(g_2(n))$.

   No hypotheses are made on the dependence of g on n.

2. Ψ is a linear function of the $m-1$ parameters $\beta, \gamma, \ldots$

   No hypotheses are made on the dependence of Ψ on n.

Then the whole system (1) is linear in the parameters $\alpha, \beta, \gamma, \ldots$, *i.e.*, if $g_i(n)$ is the solution corresponding to the values $\alpha = \alpha_i, \beta = \beta_i, \gamma = \gamma_i, \ldots$, then $g(n) = \lambda_1 g_1(n) + \lambda_2 g_2(n)$ is the solution corresponding to $\alpha = \lambda_1 \alpha_1 + \lambda_2 \alpha_2, \beta = \lambda_1 \beta_1 + \lambda_2 \beta_2, \gamma = \lambda_1 \gamma_1 + \lambda_2 \gamma_2, \ldots$

We can then look for a general solution of the form

$$g(n) = \alpha A(n) + \beta B(n) + \gamma C(n) + \ldots \tag{2}$$

i.e., think of $g(n)$ as a linear combination of m functions $A(n), B(n), C(n), \ldots$ according to the coefficients $\alpha, \beta, \gamma, \ldots$

To find these functions, we can reason as follows. Suppose we have a *repertoire* of m pairs of the form $((\alpha_i, \beta_i, \gamma_i, \ldots), g_i(n))$ satisfying the following conditions:

1. For every $i = 1, 2, \ldots, m$, $g_i(n)$ is the solution of the system corresponding to the values $\alpha = \alpha_i, \beta = \beta_i, \gamma = \gamma_i, \ldots$

2. The m m-tuples $(\alpha_i, \beta_i, \gamma_i, \ldots)$ are linearly independent.

Then the functions $A(n), B(n), C(n), \ldots$ are uniquely determined. The reason is that, for every fixed n,

$$\begin{array}{lllll} \alpha_1 A(n) & +\beta_1 B(n) & +\gamma_1 C(n) & +\ldots & = & g_1(n) \\ \vdots & & & & = & \vdots \\ \alpha_m A(n) & +\beta_m B(n) & +\gamma_m C(n) & +\ldots & = & g_m(n) \end{array}$$

is a system of m linear equations in the m unknowns $A(n), B(n), C(n), \ldots$ whose coefficients matrix is invertible.

This general idea can be applied to several different cases. For instance, if the recurrence is second-order:

$$\begin{aligned} g(0) &= \alpha_0 , \\ g(1) &= \alpha_1 , \\ g(n+1) &= \Phi_0(g(n)) + \Phi_1(g(n-1)) + \Psi(n; \beta, \gamma, \ldots) \quad \text{for } n \geq 1 , \end{aligned} \tag{3}$$

then we will require that Φ_0 and Φ_1 are linear in g, and that Ψ is a linear function of the $m-2$ parameters $\beta, \gamma, \ldots$.

The same can be said of systems of the form:

$$\begin{aligned} g(1) &= \alpha , \\ g(kn+j) &= \Phi(g(n)) + \Psi(n; \beta_j, \gamma_j, \ldots) \quad \text{for } n \geq 1 , \quad 0 \leq j < k . \end{aligned} \tag{4}$$

The previous argument is easily adapted to the new case: this time, the number of tuple-function pairs to determine will be $1 + k \cdot (m-1)$.

For instance, in the Josephus problem we have $k = 2$, $\alpha = 1$, $\Phi(g) = 2g$, $\Psi(n; \beta) = \beta$, $m = 2$, $\beta_0 = -1$, $\beta_1 = 1$: and we need $3 = 1 + 2 \cdot (2-1)$ tuple-function pairs.

Exercise A.1

Use the repertoire method to solve the following general recurrence:

$$\begin{aligned} g(0) &= \alpha , \\ g(n+1) &= 2g(n) + \beta n + \gamma \quad \text{for } n \geq 0 . \end{aligned} \tag{5}$$

Exercise A.2

What if the recurrence (5) had been

$$\begin{aligned} g(0) &= \alpha\ , \\ g(n+1) &= \delta g(n) + \beta n + \gamma \quad \text{for} \ \ n \geq 0\ . \end{aligned} \tag{6}$$

instead?

Exercise 2.21(a)

Evaluate the sum $S_n = \sum_{k=0}^{n} (-1)^{n-k}$ by the perturbation method, assuming that $n \geq 0$.