ITT9132 Concrete Mathematics

Lecture 4: 16 February 2021

Chapter Two

Sums and recurrences

Manipulation of sums

Multiple sums

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1 Sums and Recurrences
- Reduction to known solutions
- Summation factors
- Integrals

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3 Multiple sums
- Expand and contract

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Example 3: Hanoi sequence

Consider again the Tower of Hanoi recurrence:

$$T_0 = 0$$
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This sequence can be transformed into a geometric sum using the following manipulations:

- Divide both equalities by 2^n:

$$T_0/2^0 = 0$$
$$T_n/2^n = T_{n-1}/2^{n-1} + 1/2^n$$

- Set $S_n = T_n/2^n$ to have:

$$S_0 = 0$$
$$S_n = S_{n-1} + 2^{-n}$$

This is almost the geometric sum with the parameters $a = 1$ and $x = 1/2$: Only the initial summand 1 is missing.

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Example 3: Hanoi sequence

Consider again the Tower of Hanoi recurrence:

$$T_0 = 0$$
$$T_n = 2T_{n-1} + 1$$

Hence,

$$\begin{aligned} S_n &= \frac{0.5(0.5^n - 1)}{0.5 - 1} \ (a_0 = 0 \text{ has been left out of the sum}) \\ &= 1 - 2^{-n} \end{aligned}$$

$$T_n = 2^n S_n = 2^n - 1$$

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$$T_n = 2^n S_n = 2^n - 1$$

Just the same result we have proven by means of induction! :))

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Summation factor: Idea

We want to solve a linear recurrence of the form:

$$a_n T_n = b_n T_{n-1} + c_n \quad \text{for every } n > 0$$

where:

1. $\langle a_n \rangle$, $\langle b_n \rangle$ and $\langle c_n \rangle$ are arbitrary sequences; and
2. for every $n > 0$, $a_n \neq 0$ and $b_n \neq 0$.

We also assume that the *initial value* T_0 is given.

The idea

Find a summation factor s_n satisfying the following property:

$$s_n b_n = s_{n-1} a_{n-1} \quad \text{for every } n \geqslant 1$$

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Summation factor: Realization

If a sequence $\langle s_n \rangle$ as in the previous slide exists, then:

1. $s_n a_n T_n = s_n b_n T_{n-1} + s_n c_n = s_{n-1} a_{n-1} T_{n-1} + s_n c_n$.
2. Set $S_n = s_n a_n T_n$ and rewrite the equation as:

$$\begin{aligned} S_0 &= s_0 a_0 T_0 \\ S_n &= S_{n-1} + s_n c_n \end{aligned}$$

3. This yields a closed formula (!) for solution:

$$T_n = \frac{1}{s_n a_n} \left(s_0 a_0 T_0 + \sum_{k=1}^{n} s_k c_k \right) = \frac{1}{s_n a_n} \left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k \right) \text{ for every } n > 0$$

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Finding a summation factor

Assuming that $b_n \neq 0$ for every n:

1. Set $s_0 = 1$ and also $a_0 = 1$.
2. Compute the next elements using the property $s_n b_n = s_{n-1} a_{n-1}$:

$$
\begin{aligned}
s_1 &= \frac{1}{b_1} = \frac{a_0}{b_1} \\
s_2 &= \frac{s_1 a_1}{b_2} = \frac{a_0 a_1}{b_1 b_2} \\
s_3 &= \frac{s_2 a_2}{b_3} = \frac{a_0 a_1 a_2}{b_1 b_2 b_3} \\
&= \dots \\
s_n &= \frac{s_{n-1} a_{n-1}}{b_n} = \frac{a_0 a_1 \cdots a_{n-1}}{b_1 b_2 \cdots b_n}
\end{aligned}
$$

(To be proved by induction!)

Example: application of summation factor

$a_n = c_n = 1$ and $b_n = 2$ gives the Hanoi Tower sequence:

Evaluate the summation factor:

$$s_n = \frac{s_{n-1}\, a_{n-1}}{b_n} = \frac{a_0\, a_1 \cdots a_{n-1}}{b_1\, b_2 \cdots b_n} = \frac{1}{2^n}$$

The solution is:

$$T_n = \frac{1}{s_n a_n}\left(s_1\, b_1\, T_0 + \sum_{k=1}^{n} s_k\, c_k\right) = 2^n \sum_{k=1}^{n} \frac{1}{2^k} = 2^n(1 - 2^{-n}) = 2^n - 1$$

Yet Another Example: constant coefficients

Equation $Z_n = aZ_{n-1} + b$

Taking $a_n = 1$, $b_n = a$ and $c_n = b$:

- Evaluate summation factor:

$$s_n = \frac{s_{n-1}\, a_{n-1}}{b_n} = \frac{a_0\, a_1 \dots a_{n-1}}{b_1\, b_2 \dots b_n} = \frac{1}{a^n}$$

- The solution is:

$$\begin{aligned} Z_n &= \frac{1}{s_n a_n}\left(s_1\, b_1\, Z_0 + \sum_{k=1}^{n} s_k\, c_k\right) = a^n\left(Z_0 + b\sum_{k=1}^{n}\frac{1}{a^k}\right) \\ &= a^n Z_0 + b\left(1 + a + a^2 + \cdots + a^{n-1}\right) \\ &= a^n Z_0 + \frac{a^n - 1}{a - 1}\, b \end{aligned}$$

Yet Another Example: check up on results

Equation $Z_n = aZ_{n-1} + b$

$$
\begin{aligned}
Z_n &= aZ_{n-1} + b \\
&= a^2 Z_{n-2} + ab + b \\
&= a^3 Z_{n-3} + a^2 b + ab + b
\end{aligned}
$$

$\ldots\ldots$

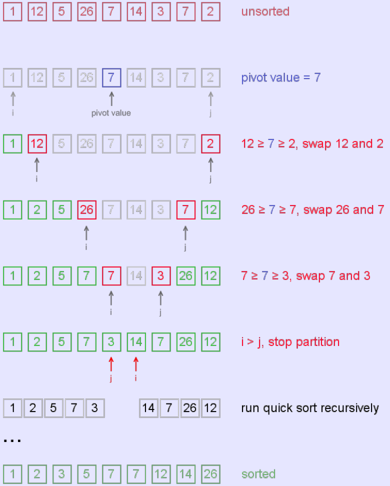
$$
\begin{aligned}
&= a^k Z_{n-k} + (a^{k-1} + a^{k-2} + \ldots + 1)b \\
&= a^k Z_{n-k} + \frac{a^k - 1}{a - 1} b \qquad \text{(assuming } a \neq 1\text{)}
\end{aligned}
$$

Continuing until $k = n$:

$$
\begin{aligned}
Z_n &= a^n Z_{n-n} + \frac{a^n - 1}{a - 1} b \\
&= a^n Z_0 + \frac{a^n - 1}{a - 1} b
\end{aligned}
$$

Efficiency of Quicksort

Average number of comparisons: $C_n = n + 1 + \frac{2}{n}\sum_{k=0}^{n-1} C_k$, $C_0 = 0$.



Efficiency of Quicksort (2)

The following transformations reduce this equation

$$nC_n = n^2 + n + 2\sum_{k=0}^{n-1} C_k$$

Write the last equation for $n-1$:

$$(n-1)C_{n-1} = (n-1)^2 + (n-1) + 2\sum_{k=0}^{n-2} C_k$$

and subtract to eliminate the sum:

$$\begin{aligned}
nC_n - (n-1)C_{n-1} &= n^2 + n + 2C_{n-1} - (n-1)^2 - (n-1) \\
nC_n - nC_{n-1} + C_{n-1} &= n^2 + n + 2C_{n-1} - n^2 + 2n - 1 - n + 1 \\
nC_n - nC_{n-1} &= C_{n-1} + 2n \\
nC_n &= (n+1)C_{n-1} + 2n
\end{aligned}$$

Efficiency of Quicksort (3)

Equation $nC_n = (n+1)C_{n-1} + 2n$

- Evaluate summation factor with $a_n = n$, $b_n = n+1$ and $c_n = 2n$:

$$s_n = \frac{a_1 a_2 \cdots a_{n-1}}{b_2 b_3 \cdots b_n} = \frac{1 \cdot 2 \cdots (n-1)}{3 \cdot 4 \cdots (n+1)} = \frac{2}{n(n+1)}$$

- Then the solution of the recurrence is:

$$\begin{aligned} C_n &= \frac{1}{s_n a_n}\left(s_1 b_1 C_0 + \sum_{k=1}^{n} s_k c_k\right) \\ &= \frac{n+1}{2} \sum_{k=1}^{n} \frac{4k}{k(k+1)} \\ &= 2(n+1) \sum_{k=1}^{n} \frac{1}{k+1} = 2(n+1)\left(\sum_{k=1}^{n} \frac{1}{k} + \frac{1}{n+1} - 1\right) \\ &= 2(n+1)H_n - 2n \end{aligned}$$

where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \approx \ln n$ is the nth harmonic number.

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A basic continuous method for discrete mathematics

To compute a sum of the form $S_n = \sum_{k=1}^{n} a_k$:

1. Choose a continuous function $f(x)$ such that $f(k) = a_k$ for every $k > 0$ integer.
2. Identify the sequence $\langle a_k \rangle$ with the staircase function

$$a(x) = \sum_{k \geqslant 1} a_k \, [k-1 < x \leqslant k]$$

3. Determine an error term E_n such that:

$$S_n = \int_0^n f(x)\, dx + E_n \text{ for every } n \geqslant 1$$

4. Express E_n itself as a sum:

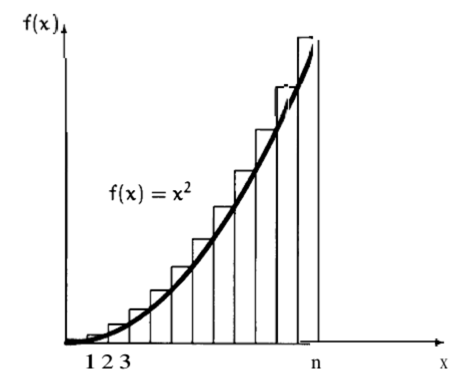
$$E_n = \sum_{k=1}^{n} \left(a_k - \int_{k-1}^{k} f(x) dx \right)$$

5. Use a closed form for E_n to determine a closed form for S_n.

Example

Example: $\square_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Replace sums by integrals.



$$\int_0^n x^2\,dx = \frac{n^3}{3} \tag{1}$$

$$\square_n = \int_0^n x^2\,dx + E_n \tag{2}$$

$$E_n = \sum_{k=1}^{n} \left(k^2 - \int_{k-1}^{k} x^2\,dx \right) \tag{3}$$

Example

Example: $\square_n = \sum_{0 \leqslant k \leqslant n} k^2$ for $n \geqslant 0$

Replace sums by integrals.

Evaluate (3):

$$
\begin{aligned}
E_n &= \sum_{k=1}^{n} \left(k^2 - \int_{k-1}^{k} x^2 \, dx \right) \\
&= \sum_{k=1}^{n} \left(k^2 - \frac{k^3 - (k-1)^3}{3} \right) \\
&= \sum_{k=1}^{n} \left(k - \frac{1}{3} \right) \\
&= \frac{(n+1)n}{2} - \frac{n}{3} = \frac{3n^2 + n}{6} .
\end{aligned}
$$

Finally, from (2) and (1) we get :

$$\square_n = \frac{n^3}{3} + \frac{3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}$$

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Manipulation of Sums

For every finite set K and permutation $p(k)$ of K:

- Distributive law:

$$\sum_{k \in K} ca_k = c \sum_{k \in K} a_k$$

- Associative law:

$$\sum_{k \in K} (a_k + b_k) = \sum_{k \in K} a_k + \sum_{k \in K} b_k$$

- Commutative law:

$$\sum_{k \in K} a_k = \sum_{p(k) \in K} a_{p(k)}$$

All of the above work because the summands are nonzero at most finitely many times. (More on this later.)

Example: Arithmetic progressions

Let's compute again:

$$S = \sum_{0 \leqslant k \leqslant n} (a + bk)$$

$$\begin{aligned}
S &= \sum_{0 \leqslant n-k \leqslant n} (a + b(n-k)) \text{ by commutativity} \\
&= \sum_{0 \leqslant k \leqslant n} (a + bn - bk) \text{ because } [0 \leqslant k \leqslant n] = [0 \leqslant n-k \leqslant n] \\
2S &= \sum_{0 \leqslant k \leqslant n} ((a + bk) + (a + bn - bk)) \text{ by associativity} \\
&= \sum_{0 \leqslant k \leqslant n} (2a + bn) \\
2S &= (2a + bn) \sum_{0 \leqslant k \leqslant n} 1 \text{ by distributivity} \\
&= (2a + bn)(n+1)
\end{aligned}$$

Again, but only using basic properties:

$$S = (n+1)a + \frac{n(n+1)}{2} b$$

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Yet Another Useful Equality

The Inclusion-Exclusion Principle

For any two finite sets K and K':

$$\sum_{k \in K} a_k + \sum_{k \in K'} a_k = \sum_{k \in K \cup K'} a_k + \sum_{k \in K \cap K'} a_k$$

Examples:

1. For $1 \leqslant m \leqslant n$:

$$\sum_{k=1}^{m} a_k + \sum_{k=m}^{n} a_k = a_m + \sum_{k=1}^{n} a_k$$

2. For $n \geqslant 0$:

$$\sum_{0 \leqslant k \leqslant n} a_k = a_0 + \sum_{1 \leqslant k \leqslant n} a_k$$

3. For $n \geqslant 0$:

$$S_n + a_{n+1} = a_0 + \sum_{0 \leqslant k \leqslant n} a_{k+1}$$

that is, we recover the perturbation method!

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Multiple sums

Definition

If H is a finite subset of $\mathbb{Z}^2$, we put:

$$\sum_{(j,k)\in H} a_{j,k} = \sum_{j,k} a_{j,k}\,[P(j,k)]$$

where $P(j,k) = (j,k) \in H$.

As only finitely many summands are nonzero, the usual properties of sums can be applied, and the following holds:

Law of interchange of order of summation

$$\sum_{j}\sum_{k} a_{j,k}\,[P(j,k)] = \sum_{j,k} a_{j,k}\,[P(j,k)] = \sum_{k}\sum_{j} a_{j,k}\,[P(j,k)]$$

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Multiple sums with independent indices

If $P(j,k) = Q(j) \wedge R(k)$, then the indices j and k are independent and the double sum can be rewritten:

$$
\begin{aligned}
\sum_{j,k} a_{j,k} &= \sum_{j,k} a_{j,k} \left([Q(j) \wedge R(k)]\right) \\
&= \sum_{j,k} a_{j,k} [Q(j)] [R(k)] \\
&= \sum_{j} [Q(j)] \sum_{k} a_{j,k} [R(k)] \quad \text{by commutativity, distributivity and associativity} \\
&= \sum_{j} \sum_{k} a_{j,k} \\
&= \sum_{k} [R(k)] \sum_{j} a_{j,k} [Q(j)] \\
&= \sum_{k} \sum_{j} a_{j,k}
\end{aligned}
$$

Multiple sums with dependent indices

In general, the indices are not independent, but we can write:

$$P(j,k) = Q(j) \wedge R'(j,k) = R(k) \wedge Q'(j,k)$$

In this case, for $K'(j) = \{k \mid R'(j,k)\}$ and $J'(k) = \{j \mid Q'(j,k)\}$ we can proceed as follows:

$$\begin{aligned}
\sum_{j,k} a_{j,k} &= \sum_{j,k} a_{j,k}[Q(j)][R'(j,k)] \\
&= \sum_{j}[Q(j)] \sum_{k} a_{j,k}[R'(j,k)] = \sum_{j \in J} \sum_{k \in K'(j)} a_{j,k} \\
&= \sum_{k}[R(k)] \sum_{j} a_{j,k}[Q'(j,k)] = \sum_{k \in K} \sum_{j \in J'(k)} a_{j,k}
\end{aligned}$$

Warmup: what's wrong with this sum?

$$\begin{aligned}
\left(\sum_{j=1}^{n} a_j\right) \cdot \left(\sum_{k=1}^{n} \frac{1}{a_k}\right) &= \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{a_j}{a_k} \\
&= \sum_{k=1}^{n} \sum_{k=1}^{n} \frac{a_k}{a_k} \\
&= \sum_{k=1}^{n} \sum_{k=1}^{n} 1 \\
&= n^2
\end{aligned}$$

Warmup: what's wrong with this sum?

$$\left(\sum_{j=1}^{n} a_j\right) \cdot \left(\sum_{k=1}^{n} \frac{1}{a_k}\right) = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{a_j}{a_k}$$
$$= \sum_{k=1}^{n} \sum_{k=1}^{n} \frac{a_k}{a_k}$$
$$= \sum_{k=1}^{n} \sum_{k=1}^{n} 1$$
$$= n^2$$

Solution

The second passage is seriously wrong:
It is not licit to turn two independent variables into two dependent ones.

Examples of multiple summing: Mutual upper bounds

Compute: $\sum_{j=1}^{n} \sum_{k=j}^{n} a_j a_k = \sum_{1 \leqslant j \leqslant n} \sum_{j \leqslant k \leqslant n} a_j a_k$.

Examples of multiple summing: Mutual upper bounds

Compute: $\sum_{j=1}^{n}\sum_{k=j}^{n} a_j a_k = \sum_{1\leqslant j\leqslant n}\sum_{j\leqslant k\leqslant n} a_j a_k$.

A crucial observation

$$[1\leqslant j\leqslant n][j\leqslant k\leqslant n] = [1\leqslant j\leqslant k\leqslant n] = [1\leqslant k\leqslant n][1\leqslant j\leqslant k]$$

Hence,

$$\sum_{j=1}^{n}\sum_{k=j}^{n} a_j a_k = \sum_{k=1}^{n}\sum_{j=1}^{k} a_j a_k$$

Also,

$$[1\leqslant j\leqslant k\leqslant n] + [1\leqslant k\leqslant j\leqslant n] = [1\leqslant j,k\leqslant n] + [1\leqslant j = k\leqslant n]$$

Examples of multiple summing: Mutual upper bounds

Compute: $\sum_{j=1}^{n} \sum_{k=j}^{n} a_j a_k = \sum_{1 \leqslant j \leqslant n} \sum_{j \leqslant k \leqslant n} a_j a_k$.

A crucial observation (cont.)

This can also be understood by considering the following matrix:

$$\begin{pmatrix} a_1 a_1 & a_1 a_2 & a_1 a_3 & \dots & a_1 a_n \\ a_2 a_1 & a_2 a_2 & a_2 a_3 & \dots & a_2 a_n \\ a_3 a_1 & a_3 a_2 & a_3 a_3 & \dots & a_2 a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n a_1 & a_n a_2 & a_n a_3 & \dots & a_n a_n \end{pmatrix}$$

and observing that $\sum_{j=1}^{n} \sum_{k=j}^{n} a_j a_k = S_U$ is the sum of the elements of the upper triangular part of the matrix.

Examples of multiple summing: Mutual upper bounds

Compute: $\sum_{j=1}^{n} \sum_{k=j}^{n} a_j a_k = \sum_{1 \leqslant j \leqslant n} \sum_{j \leqslant k \leqslant n} a_j a_k$.

A crucial observation (end)

If we add to S_U the sum $S_L = \sum_{k=1}^{n} \sum_{j=1}^{k} a_j a_k$ of the elements of the lower triangular part of the matrix, we count each element of the matrix once, except those on the main diagonal, which we count twice.

But the matrix is symmetric, so $S_U = S_L$, and

$$S_U = \frac{1}{2} \left(\left(\sum_{k=1}^{n} a_k \right)^2 + \sum_{k=1}^{n} a_k^2 \right)$$

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Examples of multiple summation

Example 1

$$\begin{aligned} S_n &= \sum_{1 \leqslant k \leqslant n} \sum_{1 \leqslant j < k} \frac{1}{k-j} \\ &= \sum_{1 \leqslant k \leqslant n} \sum_{1 \leqslant k-j < k} \frac{1}{j} \\ &= \sum_{1 \leqslant k \leqslant n} \sum_{0 < j \leqslant k-1} \frac{1}{j} \\ &= \sum_{1 \leqslant k \leqslant n} H_{k-1} \\ &= \sum_{1 \leqslant k+1 \leqslant n} H_k \\ &= \sum_{0 \leqslant k < n} H_k \end{aligned}$$

Examples of multiple summation

Example 2

$$\begin{aligned}
S_n &= \sum_{1\leqslant j\leqslant n}\sum_{j<k\leqslant n}\frac{1}{k-j} \\
&= \sum_{1\leqslant j\leqslant n}\sum_{j<k+j\leqslant n}\frac{1}{k} \\
&= \sum_{1\leqslant j\leqslant n}\sum_{0<k\leqslant n-j}\frac{1}{k} \\
&= \sum_{1\leqslant j\leqslant n} H_{n-j} \\
&= \sum_{1\leqslant n-j\leqslant n} H_j \\
&= \sum_{0\leqslant j<n} H_j
\end{aligned}$$

Examples of multiple summation

Example 3

$$
\begin{aligned}
S_n &= \sum_{1 \leqslant j < k \leqslant n} \frac{1}{k-j} \\
&= \sum_{1 \leqslant j < k+j \leqslant n} \frac{1}{k} \\
&= \sum_{1 \leqslant k \leqslant n} \sum_{1 \leqslant j \leqslant n-k} \frac{1}{k} \\
&= \sum_{1 \leqslant k \leqslant n} \frac{n-k}{k} \\
&= \sum_{1 \leqslant k \leqslant n} \frac{n}{k} - \sum_{1 \leqslant k \leqslant n} 1 \\
&= n \left(\sum_{1 \leqslant k \leqslant n} \frac{1}{k} \right) - n \\
&= nH_n - n
\end{aligned}
$$

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Another way of "simplifying by complicating"

To compute a sum of the form $S_n = \sum_{1 \leqslant k \leqslant n} a_k$:

1. Expand the summand a_k by introducing a new variable j and new summands b_j, c_k such that:
$$a_k = \sum_{1 \leqslant j \leqslant k} b_j c_k$$
2. Rewrite the sum $\sum_{1 \leqslant k \leqslant n} a_k$ as the double sum $\sum_{1 \leqslant j \leqslant k \leqslant n} b_j c_k$.
3. Contract the summands into a sum over k parameterized by j:
$$S_n = \sum_{1 \leqslant k \leqslant n} \left(\sum_{1 \leqslant j \leqslant k} b_j \right) c_k = \sum_{1 \leqslant j \leqslant n} b_j \left(\sum_{j \leqslant k \leqslant n} c_k \right)$$
4. Sum over j to obtain a closed form for S_n.

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Example: again, $\Box_n = \sum_{0 \leqslant k \leqslant n} k^2$

1. Expand: $k^2 = k \cdot k = \left(\sum_{j=1}^{k} 1\right) \cdot k$.
2. Write the double sum: $\Box_n = \sum_{1 \leqslant j \leqslant k \leqslant n} k$.
3. Contract by summing over k:

$$
\begin{aligned}
\Box_n &= \sum_{j=1}^{n} \sum_{k=j}^{n} k \\
&= \sum_{j=1}^{n} \left(\sum_{k=1}^{n} k - \sum_{k=1}^{j-1} k \right) \\
&= \sum_{j=1}^{n} \left(\frac{n(n+1)}{2} - \frac{(j-1)j}{2} \right) \\
&= \frac{1}{2} \left(n^2(n+1) - \sum_{j=1}^{n} j^2 + \sum_{j=1}^{n} j \right) \\
&= \frac{n^2(n+1)}{2} - \frac{1}{2}\Box_n + \frac{n(n+1)}{4}
\end{aligned}
$$

4. Derive a closed form for $\Box_n$:

$$\frac{3}{2}\Box_n = \frac{n+1}{4} \cdot (2n^2 + n), \text{ that is, } \Box_n = \frac{n(n+1)(2n+1)}{6}$$