Post-surjectivity and Balancedness of Cellular Automata over Groups

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Abstract. We discuss cellular automata over arbitrary finitely generated groups. We call a cellular automaton post-surjective if for any pair of asymptotic configurations, every preimage of one is asymptotic to a preimage of the other. The well known dual concept is pre-injectivity: a cellular automaton is pre-injective if distinct asymptotic configurations have distinct images. We prove that pre-injective, post-surjective cellular automata over surjunctive groups are reversible. In particular, post-surjectivity and reversibility are equivalent notions on amenable groups. We also prove that reversible cellular automata over arbitrary groups are balanced, that is, they preserve the uniform measure on the configuration space.

Keywords: cellular automata, reversibility, group theory.

1 Introduction

Cellular automata (briefly, CA) are parallel synchronous systems on regular grids where the next state of a point depends on the current state of a finite neighborhood. The grid is determined by a finitely generated group and can be visualized as the Cayley graph of the group. In addition to being a useful tool for simulations, CA also raise important and interesting questions, such as how properties of the global transition function (obtained by synchronous application of the local update rule at each point) are linked to each other.

One such relation is provided by Bartholdi's theorem [1], which links surjectivity of cellular automata to the preservation of the product measure on the space of global configurations: the latter implies the former, but is only implied

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by it if the grid satisfies additional properties. Under the same assumptions, the *Garden of Eden theorem* equates surjectivity with *pre-injectivity*, that is, the property that two asymptotic configurations (*i.e.*, two configurations differing on at most finitely many points) with the same image must be equal. In the general case, the preservation of the product measure can always be expressed combinatorially by the so-called *balancedness* property. Furthermore, bijectivity is always equivalent to reversibility, that is, the existence of an inverse that is itself a CA.

A parallel to pre-injectivity is *post-surjectivity*, which is described as follows: given a configuration e and its image c, every configuration c' asymptotic to c has a preimage e' asymptotic to e. While pre-injectivity is *weaker* than injectivity, post-surjectivity turns out to be *stronger* than surjectivity.It is natural to ask whether such trade-off between injectivity and surjectivity preserves bijectivity.

In this paper, which constitutes work in progress, we discuss the two properties above, and their links with reversibility. First, we prove that a reversible cellular automaton over any group is balanced. This gives an "almost positive" answer to a conjecture proposed in [2]. Next, we show that, in a broad setting that includes classical d-dimensional CA, post-surjectivity is equivalent to reversibility.

2 Background

If X is a set, we indicate by $\mathcal{PF}(X)$ the collection of all finite subsets of X.

Let G be a group and let $U, V \subseteq G$. We put $UV = \{x \cdot y \mid x \in U, y \in V\}$, and $U^{-1} = \{x^{-1} \mid x \in U\}$. If $U = \{g\}$ we write gV for $\{g\}V$.

A subset V of G is a *set of generators* for G if every $g \in G$ can be written as $g = w_0 \cdots w_{n-1}$ for some $w = w_0 \dots w_{n-1} \in (V \cup V^{-1})^* : G$ is *finitely generated* (briefly, f.g.) if V can be chosen finite. The *length* of $g \in G$ w.r.t. V is the *minimum* length $n = \|g\|_V$ of such a word w. The *distance* of g and h with respect to V is the length $d_V(g, h)$ of $g^{-1} \cdot h$, *i.e.*, the length of the shortest path from g to h in the *Cayley graph* of G w.r.t. V, whose vertices are the elements of G and the edges are precisely the pairs (g, gx) with $g \in G$ and $x \in V \cup V^{-1}$. The *disk* of center g and radius r w.r.t. V is the set $D_{V,r}(g)$ of those $h \in G$ such that $d_V(g, h) \leq r$: we omit g if it is the identity element 1_G of G. The function $\gamma_V(r) = |D_{V,r}|$ is the *growth rate* of G w.r.t. V. We omit V if irrelevant or clear from the context.

A group G is *amenable* if there exists a *finitely* additive probability measure μ, defined on *every* subset of G, such that $\mu(gU) = \mu(U)$ for every $g \in G$ and $U \subseteq G$. The groups $\mathbb{Z}^d$ are amenable whereas the *free groups* on two or more generators are not. For an introduction to amenability see, *e.g.*, [3, Chapter 4].

Let S be a finite set and let G be a group. The elements of the set $\mathcal{C} = S^G$ are called *configurations*. The space of configurations is given the *prodiscrete topology* by considering S as a discrete set. This makes $\mathcal{C}$ a compact space by Tychonoff's theorem. In the prodiscrete topology, two configurations are "near" if they coincide on a "large" finite subset of G: indeed, if G is f.g., then setting

$d_V(c, e) = 2^{-n}$, where n is the smallest $r \geq 0$ such that c and e differ on $D_{V,r}$, defines a distance that induces the prodiscrete topology. Two configurations are *asymptotic* if they differ at most on finitely many points of G. A *pattern* is a function $p : E \to S$ where E is a finite subset of G.

For $g \in G$, the *translation* by g is the function $\sigma_g : \mathcal{C} \to \mathcal{C}$ that sends an arbitrary configuration c into the configuration $\sigma_g(c)$ defined by

$$\sigma_g(c)(x) = c(g \cdot x) \quad \forall x \in G \,. \tag{1}$$

A *cellular automaton* (briefly, CA) on a group G is a triple $\mathcal{A} = \langle S, \mathcal{N}, f \rangle$ where the *alphabet* S is a finite set, the *neighborhood* $\mathcal{N}$ is a finite subset of G, and the *local update rule* is a function that associates to every pattern $p : \mathcal{N} \to S$ a state $f(p) \in S$. The *global transition function* of $\mathcal{A}$ is the function $F_{\mathcal{A}} : S^G \to S^G$ defined by

$$F_{\mathcal{A}}(c)(g) = f\left((\sigma_g(c))|_{\mathcal{N}} \right) \quad \forall g \in G \,: \tag{2}$$

that is, if $\mathcal{N} = \{n_1, \ldots, n_m\}$, then $F_{\mathcal{A}}(c)(g) = f\left(c(g \cdot n_1), \ldots, c(g \cdot n_m)\right)$. Observe that (2) is continuous in the prodiscrete topology and commutes with translations, *i.e.*, $F_{\mathcal{A}} \circ \sigma_g = \sigma_g \circ F_{\mathcal{A}}$ for every $g \in G$: the *Curtis-Hedlund-Lyndon theorem* states that the continuous and translation-commuting functions from $\mathcal{C}$ to itself are precisely the CA global transition functions.

We may refer to injectivity, surjectivity, etc. of $\mathcal{A}$ meaning the corresponding properties of $F_{\mathcal{A}}$. From basic facts about compact spaces follows that the inverse of the global transition function of a bijective cellular automaton $\mathcal{A}$ is itself the global transition function of some cellular automaton: we then say that $\mathcal{A}$ is *reversible*. A group G is *surjunctive* if every injective cellular automaton on G is surjective: currently, there are no known examples of non-surjunctive groups.

If G is a subgroup of Γ and $\mathcal{A} = \langle S, \mathcal{N}, f \rangle$ is a cellular automaton on G, the cellular automaton $\mathcal{A}^{\Gamma}$ *induced* by $\mathcal{A}$ on Γ has the same set of states, neighborhood, and local update rule as $\mathcal{A}$, and maps S^{Γ} (instead of S^G) into itself via $F_{\mathcal{A}^{\Gamma}}(c)(\gamma) = f\left(c(\gamma \cdot n_1), \ldots, c(\gamma \cdot n_m)\right)$ for every $\gamma \in \Gamma$. We may also say that $\mathcal{A}$ is the *restriction* of $\mathcal{A}^{\Gamma}$ to G. It is easily seen (cf. [3, Section 1.7]) that injectivity and surjectivity are preserved by both induction and restriction.

Let $\mathcal{A} = \langle S, \mathcal{N}, f \rangle$ be a CA on a group G, let $p : E \to S$ be a pattern and let $E\mathcal{N} \subseteq M \in \mathcal{PF}(G)$. A *preimage* of p on M for $\mathcal{A}$ is a pattern $q : M \to S$ such that $F_{\mathcal{A}}(c)|_E = p$ for any $c \in \mathcal{C}$ such that $c|_M = q$. An *orphan* is a pattern that has no preimage. Similarly, a configuration which is not in the image of $\mathcal{C}$ by $F_{\mathcal{A}}$ is a *garden of Eden* for $\mathcal{A}$. By a compactness argument, every garden of Eden contains an orphan. A cellular automaton $\mathcal{A}$ is *pre-injective* if every two asymptotic configurations c, e satisfying $F_{\mathcal{A}}(c) = F_{\mathcal{A}}(e)$ are equal. The *Garden of Eden theorem* (cf. [4]) states that, for CA on amenable groups, pre-injectivity is equivalent to surjectivity; on non-amenable groups, the two properties appear to be independent of each other.

3 Balancedness

Definition 1. *A cellular automaton* $\mathcal{A} = \langle S, \mathcal{N}, f \rangle$ *on a group* G *is* balanced *if it satisfies the following property: For every two finite* $E, M \subseteq G$ *such that* $E\mathcal{N} \subseteq M$*, every pattern* $p : E \to A$ *has* $|S|^{|M|-|E|}$ *preimages on* M*.*

If G is finitely generated, it is easy to see that Definition 1 is equivalent to the following property: for every $n \geq 0$ and for every $r \geq 0$ such that $\mathcal{N} \subseteq D_r$, every pattern on D_n has exactly $|S|^{\gamma(n+r)-\gamma(n)}$ preimages on D_{n+r}. In addition (cf. [2, Remark 18]) balancedness is preserved by both induction and restriction, hence, it can be determined by only checking it on the subgroup generated by the neighborhood.

Lemma 1. *Let* G *be a group, let* S *be a finite set, and let* $F, H : S^G \to S^G$ *be* CA *global transition functions.*

1. *If* F *and* H *are both balanced, then so is* $F \circ H$*.*
2. *If* F *and* $F \circ H$ *are both balanced, then so is* H*.*
3. *If* H *and* $F \circ H$ *are both balanced, and in addition* H *is reversible, then* F *is balanced.*

Proof. It is sufficient to consider the case when G is finitely generated, *e.g.*, by the union of the neighborhoods of the two CA. Let $r \geq 0$ be large enough that the next value of a point according to both F and H only depends on the current state of a neighborhood of radius r of the point.

First, suppose F and H are both balanced. Let $p : D_n \to S$: by balancedness, p has exactly $|S|^{\gamma(n+r)-\gamma(n)}$ preimages over D_{n+r} according to H. In turn, every such preimage ha $|S|^{\gamma(n+2r)-\gamma(n+r)}$ preimages over D_{n+2r} according to F, again by balancedness. All the preimages of p on D_{n+2r} by $F \circ H$ have this form, so p has $|S|^{\gamma(n+2r)-\gamma(n)}$ preimages on D_{n+2r} according to $F \circ H$. This holds for every $n \geq 0$ and $p : D_n \to S$: thus, $F \circ H$ is balanced.

Now, suppose F is balanced but H is not. Take $n \geq 0$ and $p : D_n \to S$ having $M > |S|^{\gamma(n+r)-\gamma(n)}$ preimages according to H: by balancedness of F, each of these M preimages has exactly $|S|^{\gamma(n+2r)-\gamma(n+r)}$ preimages according to F. Then p has overall $M \cdot |S|^{\gamma(n+2r)-\gamma(n+r)} > |S|^{\gamma(n+2r)-\gamma(n)}$ preimages on D_{n+2r} according to $F \circ H$, which is thus not balanced.

Finally, suppose H and $F \circ H$ are balanced and H is reversible. As the identity CA is clearly balanced, by the previous point (with H taking the role of F and H^{-1} that of H) H^{-1} is balanced. By the first point, as $F \circ H$ and H^{-1} are both balanced, so is their composition $F = F \circ H \circ H^{-1}$.

Corollary 1. *A reversible* CA *and its inverse are either both balanced or both unbalanced.*

Definition 1 states that balanced CA give at least one preimage to each pattern, thus are surjective. On amenable groups (cf. [1]) the converse is also true; on non-amenable groups (ibid.) some surjective cellular automata are not balanced. In the last section of [2], we ask ourselves the question whether *injective* cellular automata are balanced. The answer is that, at least in all cases currently known, it is so.

Theorem 1. *Reversible* CA *are balanced.*

Proof. It is not restrictive to suppose that G is finitely generated. Let $\mathcal{A} = \langle S, \mathcal{N}, f \rangle$ be a reversible cellular automaton on G and let $F = F_{\mathcal{A}}$ be its global transition function. Fix a finite set of generators V for G. For $n \geq 0$ let D_n be the disk of radius n center in the identity element of G. Let $\mathcal{N}$ be a neighborhood for both F and F^{-1}: if $r \geq 0$ is large enough that $\mathcal{N} \subseteq D_r$, then for every $c \in S^G$ the state of both $F(c)$ and $F^{-1}(c)$ on D_n is determined by the state of c in D_{n+r}.

Let $p_1, p_2 : D_n \to S$ be two patterns. It is not restrictive to suppose $n \geq r$. We exploit reversibility of F to prove that they have the same number of preimages on D_{n+r} by constructing a bijection $T_{1,2}$ between the set of the preimages of p_1 and that of the preimages of p_2. As this will hold whatever n, p_1, and p_2 are, F will be balanced.

For $i = 1, 2$ let Q_i be the set of the preimages of p_i on D_{n+r}. Given $q_1 \in Q_1$, and having fixed a state $0 \in S$, we proceed as follows:

1. First, we extend q_1 to a configuration e_1 by setting $e_1(g) = 0$ for every $g \notin D_{n+r}$.
2. Then, we apply F to e_1, obtaining c_1. By construction, $c_1|_{D_n} = p_1$.
3. Next, from c_1 we construct c_2 by replacing p_1 with p_2 inside D_n.
4. Then, we apply F^{-1} to get a new configuration e_2.
5. Finally, we call q_2 the restriction of e_2 to D_{n+r}.

Observe that $q_2 = e_2|_{D_{n+r}} \in Q_2$. This follows immediately from $\mathcal{A}$ being reversible: by construction, if we apply F to e_2, and restrict the result to D, we end up with p_2. We call $T_{1,2} : Q_1 \to Q_2$ the function computed by performing the steps from 1 to 5, and $T_{2,1} : Q_2 \to Q_1$ the one obtained by the same steps with the roles of q_1 and q_2 swapped.

Now, by construction, c_1 and c_2 coincide outside D_n, and their updates e_1 and e_2 by F^{-1} coincide outside D_{n+r}: but e_1 is 0 outside D_{n+r}, so that updating c_2 to e_2 is the same as extending q_2 with 0 outside D_{n+r}. This means that $T_{2,1}$ is the inverse of $T_{1,2}$: consequently, Q_1 and Q_2 have the same number of elements. As p_1 and p_2 are arbitrary, any two patterns on D_n have the same number of preimages on D_{n+r}. As $n \geq 0$ is also arbitrary, $\mathcal{A}$ is balanced.

Corollary 2. *Injective cellular automata over surjunctive groups are balanced.*

4 Post-surjectivity

Definition 2. *A cellular automaton is* post-surjective *if, given a configuration c and a predecessor e of c, every configuration c' asymptotic to c has a predecessor e' asymptotic to e.*

Post-surjective CA are surjective: if $f(a,\ldots,a)=b$, then we can always find a predecessor for any pattern by pasting it over the b-uniform configuration. The vice versa is not true: the xor with the right-hand neighbor is surjective, but while $\ldots 000 \ldots$ is a fixed point, $\ldots 010 \ldots$ only has preimages that take value 1 infinitely often. Also, from the Garden of Eden theorem follows

Proposition 1. *Post-surjective* CA *on amenable groups are pre-injective.*

In addition, via a reasoning similar to the one employed in [3, Section 1.7] and [2, Remark 18], we can prove

Proposition 2. *Let $\mathcal{A}=\langle S,\mathcal{N},f\rangle$ be a cellular automaton on the group G, let Γ be a group that contains G, and let $\mathcal{A}^\Gamma$ be the* CA *induced by $\mathcal{A}$ on Γ. Then $\mathcal{A}$ is post-surjective if and only if $\mathcal{A}^\Gamma$ is post-surjective.*

In particular, post-surjectivity of arbitrary CA *is equivalent to post-surjectivity on the subgroup generated by the neighborhood.*

Theorem 2. *One-dimensional post-surjective* CA *are reversible.*

Proof. For $u \in S^*$ let $u^\omega : \{k, k+1, \ldots\} \to S$ and ${}^\omega u : \{\ldots, h-2, h-1\} \to S$ be obtained by juxtaposing copies of u, without keeping information on h or k; let then ${}^\omega u^\omega = {}^\omega u u^\omega$ with $h=k$.

Suppose, for the sake of contradiction, that $\mathcal{A}=\langle S,\mathcal{N},f\rangle$ is a post-surjective one-dimensional CA which is not reversible. As it is well known (cf. [5, Theorem 7]), there exist $u,v,w \in S^*$ such that $e_u = {}^\omega u^\omega$ and $e_v = {}^\omega v^\omega$ are different and have the same image $c = {}^\omega w^\omega$. It is not restrictive to suppose $|u|=|v|=k\cdot|w|$.

By construction, the two configurations $c_{u,v} = F({}^\omega u v^\omega)$ and $c_{v,u} = F({}^\omega v u^\omega)$ are both asymptotic to c: by post-surjectivity, there exist $x,y \in S^*$ such that $e_{u,v} = {}^\omega u x v^\omega$ and $e_{v,u} = {}^\omega v y u^\omega$ satisfy $F(e_{u,v}) = F(e_{v,u}) = c$. Again, it is not restrictive to suppose that $|x|=|y|=m\cdot|u|$ for some $m \geq 1$, and that x and y start at the same point $i \in \mathbb{Z}$.

Let now consider the configuration $e' = {}^\omega u x v^N y u^\omega$: by our previous discussion, for N large enough (*e.g.*, so that x and y do not have overlapping neighborhoods) $F_\mathcal{A}(e')$ cannot help but be c. Now, recall that e_u is also a preimage of c and note that e_u and e' are asymptotic but distinct. This means that $\mathcal{A}$, which we know is surjective, is not pre-injective, contradicting the Garden of Eden theorem.

The proof of Theorem 2 depends critically on dimension 1, where CA that are injective on periodic configurations are reversible. Moreover, in our final step, we invoke the Garden of Eden theorem, which we know from [4] not to hold for CA on generic groups. Not all is lost, however: maybe, by explicitly adding the pre-injectivity requirement, we can recover Theorem 2 on more general groups? It turns out that it is so: at least, in all known cases.

Lemma 2 (cf. [6, Lemma 29]). *Let $\mathcal{A}=\langle S,\mathcal{N},f\rangle$ be a pre-injective, post-surjective* CA *on the group G. There exists a finite $M \in \mathcal{PF}(G)$ with the following property: For every pair (e,e') of asymptotic configurations, if $c = F_\mathcal{A}(e)$ and $c' = F_\mathcal{A}(e')$ disagree only on $g \in G$, then e and e' disagree at most on gM.*

Proof. It is not restrictive to suppose $1_G \in \mathcal{N}$. It is also not restrictive to suppose $g = 1_G$, the general case being recovered through commutation of $F_\mathcal{A}$ with translations.

Let $e \in \mathcal{C}$. By pre-injectivity and post-surjectivity, there are precisely $|S| - 1$ configurations e' asymptotic to e whose image c' disagree with $c = F_\mathcal{A}(e)$ at most on 1_G: let $M_e \in \mathcal{PF}(G)$ contain all the points where any of these e' differs from e, *i.e.*, $e|_{G \setminus M_e} = e'|_{G \setminus M_e}$ for all said e'. We claim that there exists a finite superset C_e of M_e with the following property: if e_1 coincides with e inside C_e, e'_1 is asymptotic to e_1, and $c_1 = F_\mathcal{A}(e_1)$ coincides with $c'_1 = F_\mathcal{A}(e'_1)$ except at most on 1_G, then e'_1 coincides with e_1 outside M_e.

To prove our claim, let C_e be a suitable finite superset of $N_e = M_e \mathcal{N} \mathcal{N}^{-1}$. Let $e_1 \in [e]_{C_e}$: let then e'_1 be asymptotic to e_1 (not necessarily equal outside C_e) and such that $c_1 = F_\mathcal{A}(e_1)$ and $c'_1 = F_\mathcal{A}(e'_1)$ coincide outside 1_G. Let now $c' : G \to S$ satisfy $c'(1_G) = c'_1(1_G)$ and $c'(x) = c(x)$ for every $x \neq 1_G$: by pre-injectivity and post-surjectivity combined, there exists a unique e' asymptotic to e such that $c' = F_\mathcal{A}(e')$, and such e' coincides with e outside M_e. But

$$e''_1(x) = \begin{cases} e'(x) & \text{if } x \in C_e \\ e_1(x) & \text{otherwise} \end{cases}$$

is also a preimage of c'_1 asymptotic to e_1: by pre-injectivity, $e''_1 = e'_1$. By construction, e''_1 agrees with e_1 outside M_e: and so does e'_1, which proves our claim.

Now, as e varies in $\mathcal{C}$, the cylinders $[e]_{C_e}$ clearly form a covering of $\mathcal{C}$: as the latter is compact, there exists $U \in \mathcal{PF}(\mathcal{C})$ such that $\bigcup_{e \in U} [e]_{C_e} = \mathcal{C}$. Then $M = \bigcup_{e \in U} M_e$ has the property required in the thesis.

Corollary 3. *Let $\mathcal{A}$ be a pre-injective, post-surjective* CA *on the group G. There exists $M \in \mathcal{PF}(G)$ with the following property: For every pair (e, e') of asymptotic configurations, if $c = F_\mathcal{A}(e)$ and $c' = F_\mathcal{A}(e')$ disagree at most on D, then e and e' disagree at most on DM.*

Theorem 3. *Let G be a* surjunctive *group. Every pre-injective, post-surjective cellular automaton $\mathcal{A} = \langle S, \mathcal{N}, f \rangle$ on G is reversible.*

Proof. By Proposition 2, it is sufficient to consider the case where G is countable.

Let $F = F_\mathcal{A}$. Let M be as in Lemma 2: we show that F has an inverse H with neighborhood $\mathcal{N} = M^{-1}$. Actually, we prove that H is a *right* inverse of F: but a right inverse of a surjective function is injective, thus H is also surjective because of surjunctivity of G, so that F is indeed the inverse of H.

To construct the local update rule $h : S^\mathcal{N} \to S$, we proceed as follows. Fix a uniform configuration u and let $v = F(u)$. Given $g \in G$ and $p : \mathcal{N} \to S$, for every $h \in G$ put

$$y_{g,p}(h) = \begin{cases} p(g^{-1}h) & \text{if } h \in g\mathcal{N} \\ v(h) & \text{otherwise} \end{cases} \tag{3}$$

that is, let $y_{g,p}$ be obtained from v by cutting away the piece with support $g\mathcal{N}$ and pasting p as a "patch" for the "hole". By post-surjectivity and pre-injectivity combined, there exists a unique $x_{g,p} \in \mathcal{C}$ asymptotic to u such that

$F(x_{g,p}) = y_{g,p}$. Let then

$$h(p) = x_{g,p}(g) \,. \tag{4}$$

Observe that (4) does *not* depend on g: if $g' = h \cdot g$, then $y_{g',p} = \sigma_h(F(x_{g,p})), = F(\sigma_h(x_{g,p}))$, so that $x_{g',p} = \sigma_h(x_{g,p})$ by pre-injectivity, and $x_{g',p}(g') = x_{g,p}(g)$.

Let now y be *any* configuration asymptotic to v such that $y|_{g\mathcal{N}} = p$, and let x the unique preimage of y asymptotic to v: we claim that $x(g) = h(p)$. To prove this, we observe that, as y and $y_{g,p}$ are both asymptotic to v, there exists a finite sequence $y_0 = y_{g,p}, y_1, \ldots, y_m = y$ such that, for every $i = 1, \ldots, m$, y_i disagrees with y_{i-1} on a single point ℓ_i, which by construction does not belong to $g\mathcal{N}$. Consider then the unique preimages x_i of y_i asymptotic to u: by Lemma 2, for every $i = 1, \ldots, m$, x_i coincides with x_{i-1} outside $\ell_i M$, which does *not* contain g as $g \in \ell_i M$ is equivalent to $\ell_i \in g\mathcal{N}$, which is not the case. As $x_0 = x_{g,p}$ because of pre-injectivity, we can conclude that $x_{g,p}(g) = h(p)$.

The argument above holds whatever the pattern $p : \mathcal{N} \to S$ is. By applying it finitely many times to arbitrary finitely many points, we determine the following fact: if y is any configuration which is asymptotic to v, then $F(H(y)) = y$. But the set of configurations asymptotic to v is dense in $\mathcal{C}$, so it follows from continuity of F and H that $F(H(y)) = y$ for every $y \in \mathcal{C}$.

Corollary 4. *A cellular automaton on an amenable group (in particular, a d-dimensional* CA*) is post-surjective if and only if it is reversible.*

5 Conclusions

We have given a little contribution to a broad research theme by examining some links between different properties of cellular automata. In particular, we have seen how reversibility can still be obtained by weakening injectivity while strengthening surjectivity. Whether other such "transfers" are possible, is a field that we believe deserving to be explored. Another interesting issue is whether post-surjective cellular automata which are not pre-injective do or do not exist: by Bartholdi's theorem, any such example would involve non-amenable groups.

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