Monads and Interaction
Lecture 2

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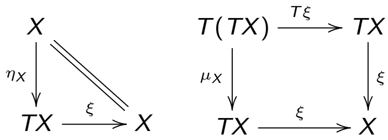
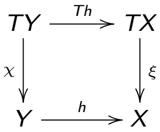
Monad algebras

Monad algebras

- An *algebra* of a monad (T, η, μ) is an object X with a map $\xi : TX \to X$ such that



- A *map* between two algebras (Y, χ) and (X, ξ) is a map h such that



- The algebras of the monad and maps between them form a category $\mathbf{Alg}(T)$, called the *Eilenberg-Moore category*, with an obvious forgetful functor $U : \mathbf{Alg}(T) \to \mathcal{C}$.

Kleisli triple algebras

- A variation of algebras fitting more smoothly with Kleisli triples is this.

- A *algebra* of a Kleisli triple $(T, \eta, (-)^\star)$ (*a Mendler-style algebra*, *an algebra in extension form*, *no-iteration form*) is given by
  - an object X,
  - a family of maps $(-)^+_Y : \mathcal{C}(Y, X) \to \mathcal{C}(TY, X)$ indexed by $Y \in |\mathcal{C}|$

  such that
  - if $f : Y \to X$, then $f^+ \circ \eta_Y = f$
  - if $k : Z \to TY$, $f : Y \to X$, then $(f^+ \circ k)^+ = f^+ \circ k^\star : TZ \to X$

- Naturality of $(-)^+$ is not required, it follows.

- There's also the correct concept of Kleisli triple algebra map. (Definition omitted.)

Monad algebras = Kleisli triple algebras

- Algebras of monads/Kleisli triples with the same carrier X are in a bijection.

- This is again crucially by the Yoneda lemma.

$$\frac{TX \to X}{\overline{\mathcal{C}(Y, X) \to \mathcal{C}(TY, X) \text{ nat. in } Y}}$$

- From ξ, one defines $(-)^+$ by $f^+ = \xi \circ Tf$.
- From $(-)^+$, one defines ξ by $\xi = \mathrm{id}_X^+$.

- The respective categories are isomorphic.

FP intuition

- An algebra of a monad T with carrier X is a "handler" of computations of values of the type X (and only of that type!).

- $\xi : TX \to X$ –

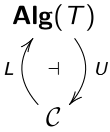
  a value of X can be extracted from a computation of values of X

- $(-)^+_Y : \mathcal{C}(Y, X) \to \mathcal{C}(TY, X)$ –

  given a way $f : Y \to X$ to "observe" values of Y as values of X, $f^+ : TY \to X$ is a way of observing computations of values of Y

Eilenberg-Moore adjunction

- In the opposite direction of $U : \mathbf{Alg}(T) \to \mathcal{C}$ there is a functor $L : \mathcal{C} \to \mathbf{Alg}(T)$ defined by
  - $LX = (TX, \mu_X)$,
  - $Lf = Tf : (TY, \mu_Y) \to (TX, \mu_X)$ for $f : Y \to X$.
- L is left adjoint to U.



$$\frac{\overbrace{(TY, \mu_Y)}^{LY} \to (X, \xi)}{Y \to \underbrace{X}_{U(X,\xi)}}$$

- This says that (TX, μ_X) is an algebra of the monad T, moreover, it is the free one.
- $U \cdot L = T$. Indeed,
  - $U(LX) = U(TX, \mu_X) = TX$,
  - if $f : Y \to X$, then $U(Lf) = U(Tf) = Tf$.
- The unit of the adjunction is η.
- The E-M resolution of a monad is its final resolution.

Algebras of exceptions monads

- Algebras of the exceptions monad $TX = E + X$ are (by definition) objects X with a map $\xi : E + X \to X$ subject to 2 equations.
- They are in a bijection with pairs of an object X and map $E \to X$.
- The E-M category of this monad is isomorphic to the coslice category $E/\mathcal{C}$.
- [FP intuition] These are handlers for exceptional computations!
- To able to extract a value from any given exceptional computation, you must know how to deal with the exception case.

Algebras of reader monads

- Algebras of the reader monad $TX = S \Rightarrow X$ are (by definition) objects X with a map $get : S \Rightarrow X \to X$ such that
  - $get\,(\lambda s.\, x) = x$
  - $get\,(\lambda s.\, get\,(\lambda s'.\, f\ s\ s')) = get\,(\lambda s.\, f\ s\ s)$

Algebras of state monads

- The E-M category of the state monad $TX = S \Rightarrow S \times X$ is isomorphic to the category of mnemoids.

- An algebra of this monad is an object X with a map $\mathit{getput} : S \Rightarrow S \times X \to X$ such that
  - $x = \mathit{getput}\,(\lambda s.\,(s, x))$
  - $\mathit{getput}\,(\lambda s.\,\mathrm{let}\ (s', g) = f\ s\ \mathrm{in}\ (s', \mathit{getput}\ g)) =$
    $\mathit{getput}\,(\lambda s.\,\mathrm{let}\ (s', g) = f\ s\ \mathrm{in}\ g\ s')$

- A *mnemoid* is an object X with maps $\mathit{get} : S \Rightarrow X \to X$ and $\mathit{put} : S \times X \to X$ such that
  - $x = \mathit{get}\,(\lambda s.\,\mathit{put}\,(s, x))$
  - $\mathit{put}\,(s, \mathit{get}\ f) = \mathit{put}\,(s, f\ s)$
  - $\mathit{put}\,(s, \mathit{put}\,(s', x)) = \mathit{put}\,(s', x)$

- From ξ, one constructs get, put by $\mathit{get}\ f = \xi\,(\lambda s.\,(s, f\ s))$, $\mathit{put}\,(s, x) = \xi\,(\lambda\_.\,(s, x))$.
- From get, put, one obtains ξ by $\xi\ f = \mathit{get}\,(\lambda s.\,\mathit{put}\,(f\ s))$.

Algebras of list monads

- The E-M category of the standard list monad is isomorphic to that of monoids,

  i.e., objects X with maps $1 \to X$ and $X \times X \to X$ satisfying left and right unitality and associativity.

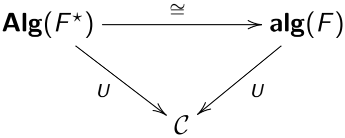
- It is therefore also called the *free monoids monad*.

- The E-M category of the alternative list monad is in a bijection with semigroups with zero.

- A *semigroup with zero* is an object X with maps $1 \to X$ and $X \times X \to X$ satisfying left and right zeroness and associativity.

Algebras of free functor-algebras monads

- The E-M category $\mathbf{Alg}(F^\star)$ of the monad $F^\star$ of free algebras of a functor F is isomorphic to the category $\mathbf{alg}(F)$ of algebras of F



- For $FX = X \times X$, algebras with carrier X of the monad F^* are maps $\mu Z.\, X + Z \times Z \to X$ subject to two equations.
- They are in bijection with algebras with carrier X of the functor F, which are maps $X \times X \to X$ subject to no conditions (*magmas*).

- A monad with this property is said to be *algebraically free* on F.

Monad maps

Monad maps

- A *monad map* between monads T, T' on a category $\mathcal{C}$ is a natural transformation $\tau : T \stackrel{\cdot}{\to} T'$ satisfying

$$
\begin{array}{ccc}
X & = & X \\
{\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta'_X} \\
TX & \xrightarrow[\tau_X]{} & T'X
\end{array}
\qquad
\begin{array}{ccccc}
T(TX) & \xrightarrow{\tau_{TX}} & T'(TX) & \xrightarrow{T'\tau_X} & T'(T'X) \\
{\scriptstyle \mu_X}\downarrow & & & & \downarrow{\scriptstyle \mu'_X} \\
TX & & \xrightarrow[\tau_X]{} & & T'X
\end{array}
$$

- Monads on $\mathcal{C}$ and maps between them form a category $\mathbf{Monad}(\mathcal{C})$.

- $\mathbf{Monad}(\mathcal{C})$ is the category of monoids in the (strict) monoidal category $([\mathcal{C}, \mathcal{C}], \mathrm{Id}_{\mathcal{C}}, \cdot)$.

Kleisli triple maps

- A map between two Kleisli triples T, T' is a family of maps $\tau_X : TX \to T'X$ indexed by $X \in |C|$ such that
  - $\tau_X \circ \eta_X = \eta'_X$,
  - if $k : X \to TY$, then $\tau_Y \circ k^\star = (\tau_Y \circ k)^{\star\prime} \circ \tau_X$.
- Naturality of τ is not required, but it follows.
- Kleisli triples on $\mathcal{C}$ and maps between them form a category isomorphic to **Monad**$(\mathcal{C})$.

Maps between exceptions, reader, writer monads

- Monad maps between the exception monads for sets E, E' are in a bijection with pairs of maps $1 \to E' + 1$ and $E \to E'$.

- Monad maps between the reader monads for sets S, S' are in a bijection with maps between S', S.

- Monad maps between the writer monads for monoids $(P, \mathrm{o}, \oplus)$ and $(P', \mathrm{o}', \oplus')$ are in a bijection with homomorphisms between these monoids.

Maps between state monads

- The monad maps between the state monads for S and S_0 are in a bijection with *(very well-behaved) lenses*.
- These are pairs of maps $coget : S_0 \to S$, $coput : S_0 \times S \to S_0$ such that
  - $s_0 = coput\,(s_0, coget\; s_0))$,
  - $coget\,(coput\,(s_0, s)) = s$,
  - $coput\,(coput\,(s_0, s), s') = coput\,(s_0, s')$.

Free functor-algebras monads are free

- The monad $F^\star$ of free algebras of a functor F (the algebraically-free monad on F), if it exists, is the free monad on F.

$$\mathbf{Monad}(\mathcal{C}) \underset{U}{\overset{(-)^\star}{\leftrightarrows}} [\mathcal{C},\mathcal{C}] \quad ((-)^\star \dashv U) \qquad\qquad \frac{F^\star \to R}{F \to UR}$$

- (Use the full subcategory of $[\mathcal{C},\mathcal{C}]$ of those functors for which $(-)^\star$ exists.)

- If a monad T is free on F, it need not be algebraically-free on F.
- A monad T is free on F iff $T \cong \mu H.\, \mathrm{Id} + F \cdot H$.
- It is algebraically free iff $TX \cong \mu Z.\, X + F(TX)$. This is generally a stronger condition.

Maps to continuation monads

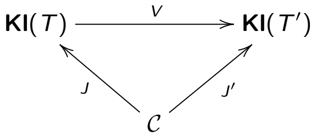
- Let xCnt^R be the external continuation monad for R ($\mathrm{xCnt}^R X = \mathcal{C}(X, R) \pitchfork R$).
- Monad maps between an arbitrary monad T and the monad xCnt^R are in a bijection with algebras of T with carrier R.
- Yoneda strikes again. :-)

$$\dfrac{\dfrac{TR \to R}{\mathcal{C}(X, R) \to \mathcal{C}(TX, R) \text{ nat. in } X}}{TX \to \mathcal{C}(X, R) \pitchfork R \text{ nat. in } X}$$

- Let Cnt^R be the continuation monad for R, which is strong.
- Strong monad maps between an arbitrary strong monad T and Cnt^R are in a bijection with algebras T with carrier R.

Monad maps vs. functors between Kleisli categories

- There is a bijection between monad maps $\tau : T \dot{\to} T'$ and functors $V : \mathbf{Kl}(T) \to \mathbf{Kl}(T')$ such that

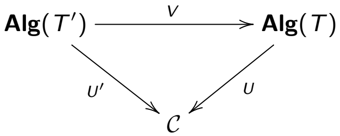


- This is defined by
  - $VX = X$,
  - $Vk = Y \xrightarrow{k} TX \xrightarrow{\tau_X} T'X$ for $k : Y \to TX$

  and
  - $\tau_X = V(TX \xrightarrow{\mathrm{id}_{TX}} {}^{T}X) : TX \to^{T'} X$.

Monad maps vs. functors between E-M categories

- There is a bijection between monad maps $\tau : T \dot{\to} T'$ and functors $V : \mathbf{Alg}(T') \to \mathbf{Alg}(T)$ such that



(Note the reversed direction.)

- This is defined by
  - $V(X, \xi) = (X, TX \xrightarrow{\tau_X} T'X \xrightarrow{\xi} X)$,
  - $Vh = h : (Y, \chi \circ \tau_Y) \to (X, \xi \circ \tau_X)$ for $h : (Y, \chi) \to (X, \xi)$

and

  - $\tau_X = \text{let } (T'X, \zeta) = V(T'X, \mu'_X) \text{ in } \zeta \circ T\eta'_X$.