Monads and interaction: Lecture 4

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Monad-comonad interaction laws

Effects happen in interaction

- To run,

  an effectful (effect-requesting) program behaving as
  a computation

  needs to **interact** with

  a environment
  that an effect-providing (coeffectful) machine behaves as

- E.g.,
  - a nondeterministic program needs a machine making choices;
  - a stateful program needs a machine coherently responding to fetch and store commands.

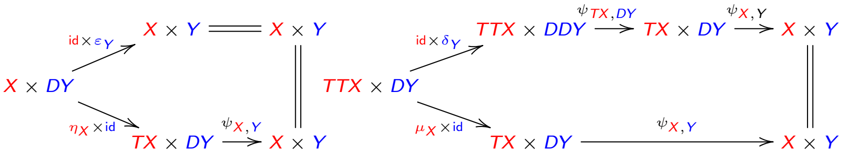
Monad-comonad interaction laws

- Let $\mathcal{C}$ be a Cartesian category. (Symmetric monoidal works too.)

- A *monad-comonad interaction law* is given by a monad (T, η, μ) and a comonad (D, ε, δ) and a nat. transf. ψ typed

$$\psi_{X,Y} : TX \times DY \to X \times Y$$

such that



- Legend:
  X – values, TX – computations
  Y – states, DY – environments (incl an initial state)

Reader monads

- $TX = S \Rightarrow X$ (the reader monad),
  $DY = S_0 \times Y$ (the coreader comonad)
  for some S_0, S and $c : S_0 \to S$

- $\psi\,(f, (s_0, y)) = (f\,(c\,s_0), y)$

- Legend:
  X – values, S – "views" of stores (data states),
  Y – (control) states, S_0 – stores (data states)

State monads

- $TX = S \Rightarrow (S \times X)$ (the state monad),
  $DY = S_0 \times (S_0 \Rightarrow Y)$ (the costate comonad)
  for some S_0, S, $c : S_0 \to S$ and $d : S_0 \times S \to S_0$
  forming a *(very well-behaved) lens*

- $\psi\,(f, (s_0, g)) = \text{let } (s', x) = f\,(c\,s_0) \text{ in } (x, g\,(d\,(s_0, s')))$

- Legend:
  X – values, S – "views" of stores (data states),
  Y – (control) states, S_0 – stores (data states)

Free functor-algebras monads (free monads)

- Free monad for intensional nondeterminism:
- $TX = \mu Z.\, X + Z \times Z$,

  $DY = \nu W.\, Y \times (W + W)$

  $\psi_{X,Y} : TX \times DY \to X \times Y$

  $\psi\,(\mathrm{in}\,(\mathrm{inl}\,x), e) = (x, \mathrm{fst}\,(\mathrm{out}\,e))$

  $\psi\,(\mathrm{in}\,(\mathrm{inr}\,(c_0, c_1)), e) = \mathrm{case}\ \mathrm{snd}\,(\mathrm{out}\,e)\ \mathrm{of} \begin{cases} \mathrm{inl}\ e' \mapsto \psi\,(c_0, e') \\ \mathrm{inr}\ e' \mapsto \psi\,(c_1, e') \end{cases}$

- Free monad for intensional store manipulation:
- $TX = \mu Z.\, X + (S \Rightarrow Z) + (S \times Z)$,

  $DY = \nu W.\, Y \times (S \times W) \times (S \Rightarrow W)$

  $\psi_{X,Y} : TX \times DY \to X \times Y$

  $\psi\,(\mathrm{in}\,(\mathrm{inl}\,x), e) = (x, \mathrm{fst}\,(\mathrm{out}\,e))$

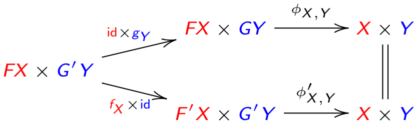
  $\psi\,(\mathrm{in}\,(\mathrm{inr}\,(\mathrm{inl}\,f)), e) = \mathrm{let}\ (s, e') = \mathrm{fst}\,(\mathrm{snd}\,(\mathrm{out}\,e))\ \mathrm{in}\ \psi\,(f\ s, e')$
  $\psi\,(\mathrm{in}\,(\mathrm{inr}\,(\mathrm{inr}\,(s, c))), e) = \psi\,(c, \mathrm{snd}\,(\mathrm{snd}\,(\mathrm{out}\,e))\ s)$

Monad-comonad interaction laws are monoids

- A *functor-functor interaction law* is given by two functors $F, G : \mathcal{C} \to \mathcal{C}$ and a family of maps

$$\phi_{X,Y} : FX \times GY \to X \times Y$$

  natural in X, Y.
- A *functor-functor interaction law map* between (F, G, ϕ), (F', G', ϕ') is given by nat. transfs. $f : F \to F'$, $g : G' \to G$ such that



- Functor-functor interaction laws form a category with a composition-based monoidal structure.
- These categories are isomorphic:
  - monad-comonad interaction laws;
  - monoid objects of the category of functor-functor interaction laws.

Some degeneracy thms for func-func int laws

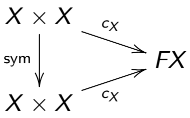
- Assume $\mathcal{C}$ is extensive ("has well-behaved coproducts").
- If F has a nullary operation, i.e., a family of maps

$$c_X : 1 \to FX$$

natural in X (eg, $F = \mathrm{Maybe}$)
or a binary commutative operation, i.e., a family of maps

$$c_X : X \times X \to FX$$

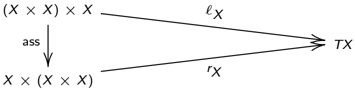
natural in X such that



(eg, $F = \mathcal{M}^{+}_{\mathrm{fin}}$) and F interacts with G, then $GY \cong 0$.

A degeneracy thm for mnd-cmnd int laws

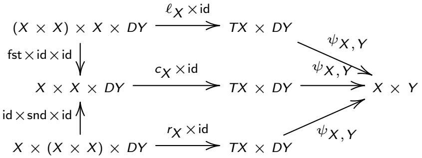
- If T has a binary associative operation, ie a family of maps $c_X : X \times X \to TX$ natural in X such that



where

$$\ell_X = (X \times X) \times X \xrightarrow{c_X \times \eta_X} TX \times TX \xrightarrow{c_{TX}} TTX \xrightarrow{\mu_X} TX$$
$$r_X = X \times (X \times X) \xrightarrow{\eta_X \times c_X} TX \times TX \xrightarrow{c_{TX}} TTX \xrightarrow{\mu_X} TX$$

(eg, $T = \mathrm{List}^+$), then any int law ψ of T and D obeys



Residual interaction laws

- Given a monad (R, η^R, μ^R) on $\mathcal{C}$.
- Eg, $R =$ Maybe, $\mathcal{M}^+$ or $\mathcal{M}$.

- A *residual functor-functor interaction law* is given by two functors $F, G : \mathcal{C} \to \mathcal{C}$ and a family of maps

$$\phi_{X,Y} : FX \times GY \to R(X \times Y)$$

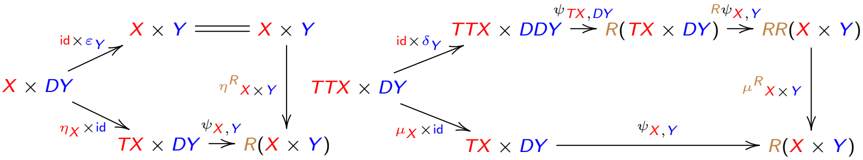
natural in X, Y.

Residual interaction laws ctd

- A *residual monad-comonad interaction law* is given by a monad (T, η, μ), a comonad (D, ε, δ) and a family of maps

$$\psi_{X,Y} : TX \times DY \to R(X \times Y)$$

natural in X, Y such that

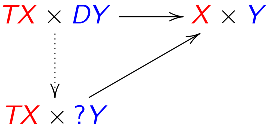


- R-residual functor-functor interaction laws form a monoidal category with R-residual monad-comonad interaction laws as monoids.

Duals

Duals

- Given a functor/monad/comonad, is there a "greatest" functor/comonad/monad interacting with it?



- The same question makes sense in the presense of a residual monad R.

Dual of a functor

- Assume again that $\mathcal{C}$ is Cartesian closed
  (or symm monoidal closed).
- For a functor $G : \mathcal{C} \to \mathcal{C}$, its *dual* is the functor $G^\circ : \mathcal{C} \to \mathcal{C}$ is

$$G^\circ X = \int_Y GY \Rightarrow (X \times Y)$$

  (if this end exists).

- $(-)^\circ$ is a functor $[\mathcal{C}, \mathcal{C}]^{\mathrm{op}} \to [\mathcal{C}, \mathcal{C}]$
  (if all functors $\mathcal{C} \to \mathcal{C}$ are dualizable;
  if not, restrict to some full subcategory of $[\mathcal{C}, \mathcal{C}]$ closed under dualization).

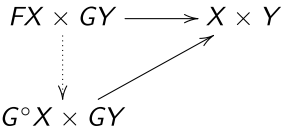
- $G^\circ = G \multimap_\star \mathrm{Id}$ where $G \multimap_\star (-)$ is the right adjoint of $(-) \star G$ and $F \star G$ is the Day convolution of F and G.

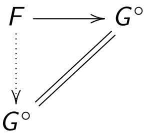
Dual of a functor ctd

- The dual G° is the "greatest" functor interacting with G.

- These categories are isomorphic:
  - functor-functor interaction laws;
  - pairs of functors F, G with nat. transfs. $F \to G^\circ$;
  - pairs of functors F, G with nat. transfs. $G \to F^\circ$.

$$\frac{FX \times GY \to X \times Y \text{ nat in } X, Y}{FX \to \underbrace{\int_Y GY \Rightarrow (X \times Y)}_{G^\circ X} \text{ nat in } X}$$





Some examples of dual

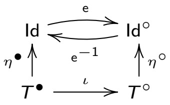
- For $GY = 0$, we have $G^\circ X \cong 1$
  and, for $GY = G_0 Y + G_1 Y$, we have $G^\circ X \cong G_0^\circ X \times G_1^\circ X$.
- For $GY = 1$, we have $G^\circ X \cong 0$.
- For $GY = A \times G' Y$, we have $G^\circ X \cong A \Rightarrow G'^\circ X$.
- For $GY = A \Rightarrow Y$, we have $G^\circ X \cong A \times X$.
- For $GY = A \Rightarrow G' Y$, we only have $G^\circ X \leftarrow A \times G'^\circ X$.
- $\mathrm{Id}^\circ \cong \mathrm{Id}$.
- But we only have $(G_0 \cdot G_1)^\circ \leftarrow G_0^\circ \cdot G_1^\circ$.
- For any G with a nullary or a binary commutative operation, we have $G^\circ X \cong 0$.

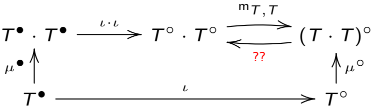
Dual of a comonad / Sweedler dual a monad

- The dual D° of a comonad D is a monad.
- This is because $(-)^\circ : [\mathcal{C}, \mathcal{C}]^{\mathrm{op}} \to [\mathcal{C}, \mathcal{C}]$ is lax monoidal, so send monoids to monoids.
- But $(-)^\circ$ is not oplax monoidal, does not send comonoids to comonoids.
- So the dual T° of a monad T is generally not a comonad.
- However we can talk about the *Sweedler dual* $T^\bullet$ of T.
- Informally, it is defined as the greatest functor D that is smaller than the functor T° and carries a comonad structure $\eta^\bullet$, $\mu^\bullet$ agreeing with η°, μ°.

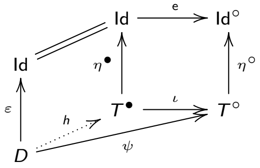
Dual of a comonad / Sweedler dual of a monad ctd

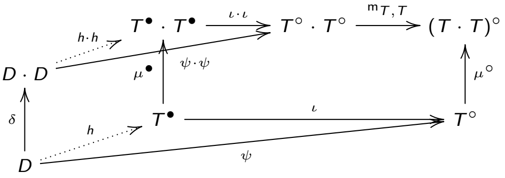
- Formally, the *Sweedler dual* of the monad T is the comonad $(T^\bullet, \eta^\bullet, \mu^\bullet)$ together with a natural transformation $\iota : T^\bullet \to T^\circ$ such that





and such that, for any comonad (D, ε, δ) together with a natural transformation ψ satisfying the same conditions, there is a unique comonad map $h : D \to T^\bullet$ satisfying





Some examples of dual and Sweedler dual

- Let $TX = \mathrm{List}^{+}X \cong \Sigma n : \mathbb{N}.\,([0..n] \Rightarrow X)$
  (the nonempty list monad) .
- We have $T^{\circ}Y \cong \Pi n : \mathbb{N}.\,([0..n] \times Y)$
  but $T^{\bullet}Y \cong Y \times (Y + Y)$.

- Let $TX = S \Rightarrow (S \times X) \cong (S \Rightarrow S) \times (S \Rightarrow X)$
  (the state monad).
- We have $T^{\circ}Y = (S \Rightarrow S) \Rightarrow (S \times Y)$
  but $T^{\bullet}Y = S \times (S \Rightarrow Y)$.

An algebraic-coalgebraic perspective

Stafeful runners

- Given
  - a resid mnd-cmnd int law, i.e., nat transf typed $\psi_{X,Y} : TX \times DY \to R(X \times Y)$ subject to eqns
  - a coEM coalgebra $(Y, \chi : Y \to DY)$ of D (a "cohandler")

  we get
  - a nat transf typed $\theta_X : TX \times Y \to R(X \times Y)$ subject to other eqns (a *resid stateful runner*)

  by

$$\theta_X = \; TX \times Y \xrightarrow{TX \times \chi} TX \times DY \xrightarrow{\psi_{X,Y}} R(X \times Y)$$

- Where do these constructions with EM (co)algebras come from?

Alternative definitions

- If $\mathcal{C}$ is Cartesian closed (or symmetric monoidal closed), R-resid mnd-cmnd int laws of T, D can be defined in multiple ways:

$$\cfrac{\cfrac{TX \times DY \to R(X \times Y) \text{ nat in } X, Y \text{ subj to eqs}}{\mathcal{C}(X \times Y, Z) \to \mathcal{C}(TX \times DY, RZ) \text{ nat in } X, Y, Z \text{ subj to eqs}}}{\cfrac{T(Y \Rightarrow Z) \to DY \Rightarrow RZ \text{ nat in } Y, Z \text{ subj to eqs}}{D(X \Rightarrow Z) \to TX \Rightarrow RZ \text{ nat in } X, Z \text{ subj to eqs}}}$$

  (Yoneda again!)

  (A symm monoidal closed category will also do.)

- Legend:
  X – values
  Y – states
  Z – observables
  (values for residual computations)
  $X \times Y \to Z$ – observation functions

A (co)algebraic view

- Resid mnd-cmnd int laws are in a bijection with coalgebra-algebra exponentiation functors:

$$T(Y \Rightarrow Z) \to DY \Rightarrow RZ \text{ nat in } Y, Z \text{ subj to eqs}$$

$$\begin{array}{ccc} (\mathbf{coEM}(D))^{\mathrm{op}} \times \mathbf{EM}(R) & \longrightarrow & \mathbf{EM}(T) \\ \Big\downarrow {\scriptstyle U^{\mathrm{op}} \times U} & & \Big\downarrow {\scriptstyle U} \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} & \xrightarrow{\Rightarrow} & \mathcal{C} \end{array}$$

$$(Y, \chi : Y \to DY), (Z, \zeta : RZ \to Z) \mapsto (Y \Rightarrow Z, T(Y \Rightarrow Z) \to (Y \Rightarrow Z))$$

$$\begin{array}{ccc} (\mathbf{coKl}(D))^{\mathrm{op}} \times \mathbf{Kl}(R) & \longrightarrow & \mathbf{EM}(T) \\ \Big\downarrow {\scriptstyle L^{D\,\mathrm{op}} \times R^{T}} & & \Big\downarrow {\scriptstyle U} \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} & \xrightarrow{\Rightarrow} & \mathcal{C} \end{array}$$

A (co)algebraic view ctd

- Explicitly, given a resid mnd-cmnd int law ψ,

  the corresponding (co)alg exp functor E sends
  an EM-coalgebra (Y, χ) of D and an EM-algebra (Z, ζ) of R to the
  EM-algebra $(Y \Rightarrow Z, \xi)$ of T where

$$\xi = \; T(Y \Rightarrow Z) \xrightarrow{\psi_{Y,Z}} DY \Rightarrow RZ \xrightarrow{\chi \Rightarrow \zeta} Y \Rightarrow Z$$

- Conversely, given a (co)alg exp functor E,

  the corresponding resid mnd-cmnd int law is

$$\psi_{Y,Z} = \; T(Y \Rightarrow Z) \xrightarrow{T(\varepsilon_Y \Rightarrow \eta^R_Z)} T(DY \Rightarrow RZ) \xrightarrow{e_{Y,Z}} DY \Rightarrow RZ$$

  where $(DY \Rightarrow RZ, e_{Y,Z}) = E((DY, \delta_Y), (RZ, \mu^R_Z))$.

Intermediate views

- In fact the picture is finer, there are also two intermediate bijections:

$$
\begin{array}{ccc}
 & \mathbf{MCIL}_R(T, D) & \\
\cong \swarrow & & \searrow \cong \\
[(\mathbf{coEM}(D))^{\mathrm{op}}, (\mathbf{SRun}_R(T))^{\mathrm{op}}]_{\mathrm{cp.}} & & [\mathbf{EM}(R), \mathbf{CRun}_D(T)]_{\mathrm{cp.}} \\
\cong \searrow & & \swarrow \cong \\
 & [(\mathbf{coEM}(D)^{\mathrm{op}} \times \mathbf{EM}(R), \mathbf{EM}(T)]_{\mathrm{ce.}} &
\end{array}
$$

where

$\mathbf{SRun}_R(T)$ - R-residual stateful runners of T
$\mathbf{CRun}_D(T)$ - D-fuelled continuation-based runners of T

Stateful runners

- For any Y, we have

R-residual stateful runners of T w/ carrier Y, ie
$TX \times Y \to R(X \times Y)$ nat in X subj to eqs

$$\overline{\overline{}}$$

monad morphisms from T to St^R_Y, ie
$TX \to Y \Rightarrow R(X \times Y)$ nat in X subj to eqs

$$\overline{\overline{}}$$

$$\begin{array}{ccc} \mathbf{EM}(R) & \longrightarrow & \mathbf{EM}(T) \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathcal{C} & \xrightarrow{Y \Rightarrow (-)} & \mathcal{C} \end{array}$$

where St^R_Y is the *R-transformed state monad* for state object Y given by

$$\mathrm{St}^R_Y X = Y \Rightarrow R(X \times Y)$$

Continuation-based runners

- For any Z, we have

D-fuelled continuation-based runners of T w/ carrier Z, ie
$D(X \Rightarrow Z) \to TX \Rightarrow Z$ nat in X subj to eqs

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monad morphisms from T to Cnt^D_Z, ie
$TX \to D(X \Rightarrow Z) \Rightarrow Z$ nat in X subj to eqs

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$$\begin{array}{ccc} (\mathbf{coEM}(D))^{\mathrm{op}} & \longrightarrow & \mathbf{EM}(T) \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathcal{C} & \xrightarrow{(-)\Rightarrow Z} & \mathcal{C} \end{array}$$

where Cnt^D_Z is the *D-transformed continuation monad* for answer object Z given by

$$\mathsf{Cnt}^D_Z X = D(X \Rightarrow Z) \Rightarrow Z$$

EM algebras of T w/ carrier $Y \Rightarrow Z$ as runners

- For any Y, Z, we have

$$\frac{\dfrac{\text{state and continuation based runners of } T \text{ w/ carrier } Z\text{, ie} \quad \mathcal{C}(X \times Y, Z) \to \mathcal{C}(TX \times Y, Z) \text{ nat in } X \text{ subj to eqs}}{\text{monad morphisms from } T \text{ to } \mathrm{xCntSt}_{Y,Z} \cong \mathrm{xCostCnt}_{Y,Z}\text{, ie} \quad TX \to Y \Rightarrow \mathrm{xCnt}_Z(X \times Y) \cong \mathrm{xCost}_Y(X \Rightarrow Z) \Rightarrow Z \text{ nat in } X \text{ subj to eqs}}}{\text{EM algebras of } T \text{ with carrier } Y \Rightarrow Z}$$

where

$$\begin{array}{rcl}
\mathrm{xCnt}_Z X & = & \mathcal{C}(X, Z) \pitchfork Z \\
\mathrm{xCntSt}_{Y,Z} X & = & Y \Rightarrow \mathrm{xCnt}_Z(X \times Y) \\
 & = & Y \Rightarrow (\mathcal{C}(X \times Y, Z) \pitchfork Z) \\
\mathrm{xCost}_Y X & = & \mathcal{C}(Y, X) \bullet Y \\
\mathrm{xCostCnt}_{Y,Z} X & = & \mathrm{xCost}_Y(X \Rightarrow Z) \Rightarrow Z \\
 & = & (\mathcal{C}(Y, X \Rightarrow Z) \bullet Y) \Rightarrow Z
\end{array}$$

Monoid-comonoid interaction laws

Residual interaction laws and Chu spaces

- The *Day convolution* of F, G is

$$(F \star G)Z = \int^{X,Y} \mathcal{C}(X \times Y, Z) \bullet (FX \times GY)$$

(if this coend exists).

- These categories are isomorphic for a given monad R:
  - R-residual functor-functor interaction laws;
  - Chu spaces on the symm monoidal category $([\mathcal{C}, \mathcal{C}], J, \star)$ with vertex R, ie, triples of two functors F, G with a nat transf $F \star G \to R$.

(if $\star$ is defined for all functors).

$$\frac{\dfrac{FX \times GY \to R(X \times Y) \text{ nat in } X, Y}{\mathcal{C}(X \times Y, Z) \to \mathcal{C}(FX \times GY, RZ) \text{ nat in } X, Y, Z}}{\underbrace{\int^{X,Y} \mathcal{C}(X \times Y, Z) \bullet (FX \times GY)}_{(F \star G)Z} \to RZ \text{ nat in } Z}$$

Residual interaction laws and Chu spaces ctd

- We do not immediately get another chacterization of the category of R-residual monad-comonad interaction laws.
- We have to use that $[\mathcal{C}, \mathcal{C}]$ has a *duoidal* structure $(\mathsf{Id}, \cdot, J, \star)$.
- In particular, $\star$ is oplax monoidal wrt $(\mathsf{Id}, \cdot)$, so there are structural laws
$$\mathsf{Id} \star \mathsf{Id} \to \mathsf{Id}$$
$$(F \cdot F') \star (G \cdot G') \to (F \star G) \cdot (F' \star G')$$
with the requisite properties.
- This duoidal structure induces a monoidal structure on $\mathbf{Chu}(R)$ based on $(\mathsf{Id}, \cdot)$.
- R-residual monad-comonad interaction laws are monoid objects of $\mathbf{Chu}(R)$ wrt this monoidal structure.

General residual interaction laws

- Instead of an endofunctor category, one can consider *any* duoidal category $(\mathcal{D}, I, \diamond, J, \star)$.
- Given a monoid object (R, η^R, μ^R) wrt. $(I, \diamond)$, we get a $(I, \diamond)$-based monoidal structure on $\mathbf{Chu}(R)$.
- An R-residual monoid-comonoid interaction law is a monoid object of $\mathbf{Chu}(R)$.
- Explicitly, it is given by a monoid (T, η, μ), a comonoid (D, ε, δ) and a map $\psi : T \star D \to R$ such that