Monads and interaction:
Lecture 4

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MGS 2021, Sheffield, UK, 12–16 Apr. 2021
Monad-comonad interaction laws
Effects happen in interaction

- To run,
  - an effectful (effect-requesting) program behaving as a computation
    needs to interact with
    - a environment
      that an effect-providing (coeffectful) machine behaves as

- E.g.,
  - a nondeterministic program needs a machine making choices;
  - a stateful program needs a machine coherently responding to fetch and store commands.
Monad-comonad interaction laws

- Let $C$ be a Cartesian category. (Symmetric monoidal works too.)

- A *monad-comonad interaction law* is given by a monad $(T, \eta, \mu)$ and a comonad $(D, \varepsilon, \delta)$ and a nat. transf. $\psi$ typed

  $$\psi_{X,Y} : TX \times DY \to X \times Y$$

  such that

  $\xymatrix{ X \times Y \ar[r]^{id \times \varepsilon_Y} & X \times Y \ar@{=}[r] & X \times Y \ar[r]^{\eta_X \times id} & TX \times DY \ar[r]^{\psi_{X,Y}} & X \times Y \ar[r]^{id \times \delta_Y} & TTX \times DDY \ar[r]^{\psi_{TX,DY}} & TX \times DY \ar[r]^{\psi_{X,Y}} & X \times Y}$

- Legend:
  - $X$ – values, $TX$ – computations
  - $Y$ – states, $DY$ – environments (incl an initial state)
Reader monads

- $TX = S \Rightarrow X$ (the reader monad),
  $DY = S_0 \times Y$ (the coreader comonad)
  for some $S_0, S$ and $c : S_0 \to S$

- $\psi(f, (s_0, y)) = (f(c \cdot s_0), y)$

Legend:
- $X$ – values, $S$ – “views” of stores (data states),
- $Y$ – (control) states, $S_0$ – stores (data states)
State monads

- \( TX = S \Rightarrow (S \times X) \) (the state monad),
- \( DY = S_0 \times (S_0 \Rightarrow Y) \) (the costate comonad)

for some \( S_0, S, c : S_0 \to S \) and \( d : S_0 \times S \to S_0 \)
forming a \((\text{very well-behaved}) \) lens

- \( \psi(f, (s_0, g)) = \text{let } (s', x) = f(c s_0) \text{ in } (x, g(d(s_0, s'))) \)

Legend:
- \( X \) – values, \( S \) – “views” of stores (data states),
- \( Y \) – (control) states, \( S_0 \) – stores (data states)
Free functor-algebras monads (free monads)

- Free monad for intensional nondeterminism:
  \[ TX = \mu Z. X + Z \times Z, \]
  \[ DY = \nu W. Y \times (W + W) \]
  \[ \psi_{X,Y} : TX \times DY \rightarrow X \times Y \]
  \[ \psi (\text{in} (\text{inl} \ x), e) = (x, \text{fst} (\text{out} e)) \]
  \[ \psi (\text{in} (\text{inr} (c_0, c_1)), e) = \text{case } \text{snd} (\text{out} e) \text{ of } \begin{cases} \text{inl } e' \mapsto \psi (c_0, e') \\ \text{inr } e' \mapsto \psi (c_1, e') \end{cases} \]

- Free monad for intensional store manipulation:
  \[ TX = \mu Z. X + (S \Rightarrow Z) + (S \times Z), \]
  \[ DY = \nu W. Y \times (S \times W) \times (S \Rightarrow W) \]
  \[ \psi_{X,Y} : TX \times DY \rightarrow X \times Y \]
  \[ \psi (\text{in} (\text{inl} \ x), e) = (x, \text{fst} (\text{out} e)) \]
  \[ \psi (\text{in} (\text{inr} (\text{inl} f)), e) = \text{let } (s, e') = \text{fst} (\text{snd} (\text{out} e)) \text{ in } \psi (f s, e') \]
  \[ \psi (\text{in} (\text{inr} (\text{inr} (s, c))), e) = \psi (c, \text{snd} (\text{snd} (\text{out} e)) s) \]
Monad-comonad interaction laws are monoids

- A functor-functor interaction law is given by two functors $F, G : C \to C$ and a family of maps $\phi_{X,Y} : FX \times GY \to X \times Y$ natural in $X, Y$.

- A functor-functor interaction law map between $(F, G, \phi), (F', G', \phi')$ is given by nat. transfs. $f : F \to F', g : G' \to G$ such that

  $\phi_{X,Y} : FX \times GY \to X \times Y$

  $id \times g_Y : FX \times GY \to FX \times G'Y$

  $f_X \times id : F'X \times G'Y \to FX \times G'Y$

  $\phi'_{X,Y} : F'X \times G'Y \to X \times Y$

- Functor-functor interaction laws form a category with a composition-based monoidal structure.

- These categories are isomorphic:
  - monad-comonad interaction laws;
  - monoid objects of the category of functor-functor interaction laws.
Some degeneracy thms for func-func int laws

- Assume $C$ is extensive (“has well-behaved coproducts”).
- If $F$ has a nullary operation, i.e., a family of maps
  \[ c_X : 1 \to FX \]
  natural in $X$ (eg, $F = \text{Maybe}$)
  or a binary commutative operation, i.e., a family of maps
  \[ c_X : X \times X \to FX \]
  natural in $X$ such that
  \[
  \begin{array}{ccc}
  X \times X & \xrightarrow{c_X} & FX \\
  \text{sym} \downarrow & & \downarrow \\
  X \times X & \xrightarrow{c_X} & FX
  \end{array}
  \]
  (eg, $F = \mathcal{M}^+_{\text{fin}}$) and $F$ interacts with $G$, then $GY \cong 0$. 

A degeneracy thm for mnd-cmnd int laws

- If $T$ has a binary associative operation, ie a family of maps $c_X : X \times X \to TX$ natural in $X$ such that

$$
\begin{align*}
(X \times X) \times X & \xrightarrow{\text{ass}} X \times (X \times X) \\
& \xrightarrow{\ell_X} TX \\
& \xrightarrow{r_X} TX
\end{align*}
$$

where

$$
\ell_X = (X \times X) \times X \xrightarrow{c_X \times \eta_X} TX \times TX \xrightarrow{c_{TX}} TTX \xrightarrow{\mu_X} TX
$$

$$
r_X = X \times (X \times X) \xrightarrow{\eta_X \times c_X} TX \times TX \xrightarrow{c_{TX}} TTX \xrightarrow{\mu_X} TX
$$

(eg, $T = \text{List}^+$), then any int law $\psi$ of $T$ and $D$ obeys

$$
\begin{align*}
(X \times X) \times X \times DY & \xrightarrow{\ell_X \times \text{id}} TX \times DY \\
& \xrightarrow{\psi_{X,Y}} X \times Y \\
& \xrightarrow{\psi_{X,Y}} X \times Y
\end{align*}
$$
Residual interaction laws

- Given a monad \((R, \eta^R, \mu^R)\) on \(\mathcal{C}\).
- Eg, \(R = \text{Maybe}, \mathcal{M}^+\) or \(\mathcal{M}\).

A *residual functor-functor interaction law* is given by two functors \(F, G : \mathcal{C} \to \mathcal{C}\) and a family of maps

\[
\phi_{X,Y} : FX \times GY \to R(X \times Y)
\]

natural in \(X, Y\).
Residual interaction laws ctd

- A residual monad-comonad interaction law is given by a monad \((T, \eta, \mu)\), a comonad \((D, \varepsilon, \delta)\) and a family of maps 

  \[ \psi_{X,Y} : TX \times DY \to R(X \times Y) \]

  natural in \(X, Y\) such that

\[ \begin{align*}
  \text{id} \times \varepsilon_Y & : X \times Y \to X \times Y \\
  \eta^{R_{X \times Y}} & : T \times D_Y \to R(T \times D_Y) \\
  \eta^{X \times \text{id}} & : TX \times DY \to R(X \times Y) \\
  \psi_{X,Y} & : TX \times DY \to R(X \times Y) \\
  \mu^{R_{X \times Y}} & : R(T \times D_Y) \to RR(X \times Y)
\end{align*} \]

\[ \begin{align*}
  \text{id} \times \delta_Y & : T \times D_Y \to R(T \times D_Y) \\
  \mu^{X \times \text{id}} & : TX \times DY \to R(X \times Y) \\
  \psi_{X,Y} & : TX \times DY \to R(X \times Y)
\end{align*} \]

- \(R\)-residual functor-functor interaction laws form a monoidal category with \(R\)-residual monad-comonad interaction laws as monoids.
Duals
Duals

Given a functor/monad/comonad, is there a “greatest” functor/comonad/monad interacting with it?

\[ TX \times DY \rightarrow X \times Y \]

The same question makes sense in the presence of a residual monad \( R \).
**Dual of a functor**

- Assume again that $\mathcal{C}$ is Cartesian closed (or symm monoidal closed).
- For a functor $G : \mathcal{C} \to \mathcal{C}$, its *dual* is the functor $G^\circ : \mathcal{C} \to \mathcal{C}$ is
  \[ G^\circ X = \int_Y G Y \Rightarrow (X \times Y) \]
  (if this end exists).

- $(-)^\circ$ is a functor $[C, C]^{op} \to [C, C]$
  (if all functors $C \to C$ are dualizable; if not, restrict to some full subcategory of $[C, C]$ closed under dualization).

- $G^\circ = G \ast \text{Id}$ where $G \ast (-)$ is the right adjoint of $(-) \ast G$ and $F \ast G$ is the Day convolution of $F$ and $G$. 
Dual of a functor ctd

- The dual $G^\circ$ is the “greatest” functor interacting with $G$.

- These categories are isomorphic:
  - functor-functor interaction laws;
  - pairs of functors $F$, $G$ with nat. transfs. $F \rightarrow G^\circ$;
  - pairs of functors $F$, $G$ with nat. transfs. $G \rightarrow F^\circ$.

\[
\begin{align*}
FX \times GY &\rightarrow X \times Y \text{ nat in } X, Y \\
FX &\rightarrow \int_Y GY \Rightarrow (X \times Y) \text{ nat in } X \\
&\quad \underbrace{G^\circ X}
\end{align*}
\]

\[
\begin{align*}
FX \times GY &\rightarrow X \times Y \\
G^\circ X \times GY &\rightarrow \quad F \rightarrow G^\circ
\end{align*}
\]
Some examples of dual

- For $GY = 0$, we have $G^\circ X \cong 1$
  and, for $GY = G_0 Y + G_1 Y$, we have $G^\circ X \cong G_0^\circ X \times G_1^\circ X$.

- For $GY = 1$, we have $G^\circ X \cong 0$.
- For $GY = A \times G' Y$, we have $G^\circ X \cong A \Rightarrow G'^\circ X$.

- For $GY = A \Rightarrow Y$, we have $G^\circ X \cong A \times X$.
- For $GY = A \Rightarrow G' Y$, we only have $G^\circ X \leftarrow A \times G'^\circ X$.

- $\text{Id}^\circ \cong \text{Id}$.

- But we only have $(G_0 \cdot G_1)^\circ \leftarrow G_0^\circ \cdot G_1^\circ$.

- For any $G$ with a nullary or a binary commutative operation, we have $G^\circ X \cong 0$. 
Dual of a comonad / Sweedler dual a monad

- The dual $D^\circ$ of a comonad $D$ is a monad.
- This is because $(-)^\circ : [C, C]^{op} \to [C, C]$ is lax monoidal, so send monoids to monoids.

- But $(-)^\circ$ is not oplax monoidal, does not send comonoids to comonoids.
- So the dual $T^\circ$ of a monad $T$ is generally not a comonad.

- However we can talk about the _Sweedler dual_ $T^\bullet$ of $T$.
- Informally, it is defined as the greatest functor $D$ that is smaller than the functor $T^\circ$ and carries a comonad structure $\eta^\bullet$, $\mu^\bullet$ agreeing with $\eta^\circ$, $\mu^\circ$. 
Formally, the *Sweedler dual* of the monad \( T \) is the comonad \((T^\bullet, \eta^\bullet, \mu^\bullet)\) together with a natural transformation \( \iota : T^\bullet \to T^\circ \) such that

\[
\begin{align*}
\text{Id} & \xrightarrow{e} \text{Id}^\circ \\
\eta^\bullet & \xrightarrow{e^{-1}} \eta^\circ \\
T^\bullet & \xrightarrow{\iota} T^\circ \\
\end{align*}
\]

and such that, for any comonad \((D, \varepsilon, \delta)\) together with a natural transformation \( \psi \) satisfying the same conditions, there is a unique comonad map \( h : D \to T^\bullet \) satisfying

\[
\begin{align*}
\text{Id} & \xrightarrow{e} \text{Id}^\circ \\
\eta^\bullet & \xrightarrow{e^{-1}} \eta^\circ \\
T^\bullet & \xrightarrow{\iota} T^\circ \\
\end{align*}
\]
Some examples of dual and Sweedler dual

- Let $TX = \text{List}^+ X \cong \Sigma n : \mathbb{N} \cdot ([0..n] \Rightarrow X)$ (the nonempty list monad).

- We have $T^\circ Y \cong \prod n : \mathbb{N} \cdot ([0..n] \times Y)$ but $T^\bullet Y \cong Y \times (Y + Y)$.

- Let $TX = S \Rightarrow (S \times X) \cong (S \Rightarrow S) \times (S \Rightarrow X)$ (the state monad).

- We have $T^\circ Y = (S \Rightarrow S) \Rightarrow (S \times Y)$ but $T^\bullet Y = S \times (S \Rightarrow Y)$. 
An algebraic-coalgebraic perspective
Stafeful runners

- Given
  - a resid mnd-cmnd int law, i.e., nat transf typed $\psi_{X,Y} : TX \times DY \to R(X \times Y)$ subject to eqns
  - a coEM coalgebra $(Y, \chi : Y \to DY)$ of $D$ (a “cohandler”)

we get

- a nat transf typed $\theta_X : TX \times Y \to R(X \times Y)$ subject to other eqns (a resid stateful runner)

by

$$
\theta_X = \begin{array}{c}
TX \times Y \\
\xrightarrow{TX \times \chi} \\
TX \times DY \\
\xrightarrow{\psi_{X,Y}} \\
R(X \times Y)
\end{array}
$$

Where do these constructions with EM (co)algebras come from?
Alternative definitions

- If \( C \) is Cartesian closed (or symmetric monoidal closed), \( R \)-resid mnd-cmnd int laws of \( T, D \) can be defined in multiple ways:

\[
\begin{align*}
TX \times DY &\to R(X \times Y) \text{ nat in } X, Y \text{ subj to eqs} \\
C(X \times Y, Z) &\to C(TX \times DY, RZ) \text{ nat in } X, Y, Z \text{ subj to eqs} \\
T(Y \Rightarrow Z) &\to DY \Rightarrow RZ \text{ nat in } Y, Z \text{ subj to eqs} \\
D(X \Rightarrow Z) &\to TX \Rightarrow RZ \text{ nat in } X, Z \text{ subj to eqs}
\end{align*}
\]

(Yoneda again!)

(A symm monoidal closed category will also do.)

- Legend:
  
  - \( X \) – values
  - \( Y \) – states
  - \( Z \) – observables
  - (values for residual computations)
  - \( X \times Y \to Z \) – observation functions
A (co)algebraic view

- Resid mnd-cmnd int laws are in a bijection with coalgebra-algebra exponentiation functors:

\[ T(Y \Rightarrow Z) \rightarrow DY \Rightarrow RZ \text{ nat in } Y, Z \text{ subj to eqs} \]

\[
\begin{array}{c}
\text{(coEM(D))}^{\text{op}} \times \text{EM(R)} \longrightarrow \text{EM(T)} \\
\downarrow U^{\text{op}} \times U \\
C^{\text{op}} \times C \Rightarrow C \\
\end{array}
\]

\[
(Y, \chi : Y \to DY), (Z, \zeta : RZ \to Z) \mapsto (Y \Rightarrow Z, T(Y \Rightarrow Z) \to (Y \Rightarrow Z))
\]

\[
\begin{array}{c}
\text{(coKI(D))}^{\text{op}} \times \text{KI(R)} \longrightarrow \text{EM(T)} \\
\downarrow L^{D^{\text{op}}} \times R^T \\
C^{\text{op}} \times C \Rightarrow C \\
\end{array}
\]
A (co)algebraic view ctd

- Explicitly, given a resid mnd-cmnd int law $\psi$, the corresponding (co)alg exp functor $E$ sends an EM-coalgebra $(Y, \chi)$ of $D$ and an EM-algebra $(Z, \zeta)$ of $R$ to the EM-algebra $(Y \Rightarrow Z, \xi)$ of $T$ where

$$
\xi = T(Y \Rightarrow Z) \xrightarrow{T(\varepsilon_Y \Rightarrow \eta^R_Z)} DY \Rightarrow RZ \xrightarrow{\chi \Rightarrow \zeta} Y \Rightarrow Z
$$

- Conversely, given a (co)alg exp functor $E$, the corresponding resid mnd-cmnd int law is

$$
\psi_{Y,Z} = T(Y \Rightarrow Z) \xrightarrow{T(\varepsilon_Y \Rightarrow \eta^R_Z)} T(DY \Rightarrow RZ) \xrightarrow{e_Y, Z} DY \Rightarrow RZ
$$

where $(DY \Rightarrow RZ, e_Y, Z) = E((DY, \delta_Y), (RZ, \mu^R_Z))$. 
Intermediate views

- In fact the picture is finer, there are also two intermediate bijections:

\[
\begin{align*}
\text{MCIL}_R(T, D) & \cong \left[ (\text{coEM}(D))^{\text{op}}, (\text{SRun}_R(T))^{\text{op}} \right]_{\text{cp}} \cong \left[ \text{EM}(R), \text{CRun}_D(T) \right]_{\text{cp}} \\
& \cong \left[ (\text{coEM}(D)^{\text{op}} \times \text{EM}(R), \text{EM}(T)) \right]_{\text{ce}}
\end{align*}
\]

where

- \(\text{SRun}_R(T)\) - \(R\)-residual stateful runners of \(T\)
- \(\text{CRun}_D(T)\) - \(D\)-fuelled continuation-based runners of \(T\)
Stateful runners

- For any \( Y \), we have

  \[
  R\text{-residual stateful runners of } T \text{ w/ carrier } Y, \text{ ie } \quad TX \times Y \to R(X \times Y) \text{ nat in } X \text{ subj to eqs}
  \]

  \[
  \text{monad morphisms from } T \text{ to } St^R_Y, \text{ ie } \quad TX \to Y \Rightarrow R(X \times Y) \text{ nat in } X \text{ subj to eqs}
  \]

\[
\begin{array}{ccc}
\text{EM}(R) & \longrightarrow & \text{EM}(T) \\
\uparrow U & & \downarrow U \\
C & \stackrel{Y \Rightarrow (-)}{\longrightarrow} & C
\end{array}
\]

where \( St^R_Y \) is the \( R\text{-transformed state monad} \) for state object \( Y \) given by

\[
St^R_Y X = Y \Rightarrow R(X \times Y)
\]
Continuation-based runners

- For any $Z$, we have

\[
D\text{-fuelled continuation-based runners of } T \text{ w/ carrier } Z, \text{ ie } \\
D(X \Rightarrow Z) \rightarrow TX \Rightarrow Z \text{ nat in } X \text{ subj to eqs} \\
\text{monad morphisms from } T \text{ to } \text{Cnt}^D_Z, \text{ ie } \\
TX \rightarrow D(X \Rightarrow Z) \Rightarrow Z \text{ nat in } X \text{ subj to eqs} \\
(\text{coEM}(D))^{\text{op}} \longrightarrow \text{EM}(T) \\
\text{where } \text{Cnt}^D_Z \text{ is the } D\text{-transformed continuation monad} \text{ for answer object } Z \text{ given by} \\
\text{Cnt}^D_Z X = D(X \Rightarrow Z) \Rightarrow Z
\]
EM algebras of $T$ w/ carrier $Y \Rightarrow Z$ as runners

- For any $Y, Z$, we have

  
  state and continuation based runners of $T$ w/ carrier $Z$, ie
  
  \[ C(X \times Y, Z) \rightarrow C(TX \times Y, Z) \] nat in $X$ subj to eqs

  
  monad morphisms from $T$ to $xCntSt_{Y,Z} \cong xCostCnt_{Y,Z}$, ie
  
  \[ TX \rightarrow Y \Rightarrow xCnt_Z(X \times Y) \]
  
  \[ \cong xCost_Y(X \Rightarrow Z) \Rightarrow Z \] nat in $X$ subj to eqs

  
  EM algebras of $T$ with carrier $Y \Rightarrow Z$

where

\[
\begin{align*}
  xCnt_Z X &= C(X, Z) \triangleleft Z \\
  xCntSt_{Y,Z} X &= Y \Rightarrow xCnt_Z(X \times Y) \\
  &= Y \Rightarrow (C(X \times Y, Z) \triangleleft Z) \\
  xCost_Y X &= C(Y, X) \bullet Y \\
  xCostCnt_{Y,Z} X &= xCost_Y(X \Rightarrow Z) \Rightarrow Z \\
  &= (C(Y, X \Rightarrow Z) \bullet Y) \Rightarrow Z
\end{align*}
\]
Monoid-comonoid interaction laws
Residual interaction laws and Chu spaces

- The *Day convolution* of $F, G$ is

$$ (F \star G)Z = \int^{X, Y} C(X \times Y, Z) \bullet (FX \times GY) $$

(if this coend exists).

- These categories are isomorphic for a given monad $R$:
  - $R$-residual functor-functor interaction laws;
  - Chu spaces on the symm monoidal category $([C, C], J, \ast)$ with vertex $R$, i.e., triples of two functors $F, G$ with a nat transf $F \star G \to R$.

(if $\ast$ is defined for all functors).

\[
\begin{align*}
FX \times GY & \to R(X \times Y) \text{ nat in } X, Y \\
C(X \times Y, Z) & \to C(FX \times GY, RZ) \text{ nat in } X, Y, Z \\
\int^{X, Y} C(X \times Y, Z) \bullet (FX \times GY) & \to RZ \text{ nat in } Z \\
(F \star G)Z & \\
\end{align*}
\]
We do not immediately get another characterization of the category of $R$-residual monad-comonad interaction laws.

We have to use that $[C, C]$ has a *duoidal* structure $(\text{Id}, \cdot, J, \star)$.

In particular, $\star$ is oplax monoidal wrt $(\text{Id}, \cdot)$, so there are structural laws

$$\text{Id} \star \text{Id} \rightarrow \text{Id}$$

$$(F \cdot F') \star (G \cdot G') \rightarrow (F \star G) \cdot (F' \star G')$$

with the requisite properties.

This duoidal structure induces a monoidal structure on $\text{Chu}(R)$ based on $(\text{Id}, \cdot)$.

$R$-residual monad-comonad interaction laws are monoid objects of $\text{Chu}(R)$ wrt this monoidal structure.
General residual interaction laws

- Instead of an endofunctor category, one can consider any duoidal category \((\mathcal{D}, I, \diamond, J, \ast)\).
- Given a monoid object \((R, \eta^R, \mu^R)\) wrt. \((I, \diamond)\), we get a \((I, \diamond)\)-based monoidal structure on \(\text{Chu}(R)\).
- An \(R\)-residual monoid-comonoid interaction law is a monoid object of \(\text{Chu}(R)\).
- Explicitly, it is given by a monoid \((T, \eta, \mu)\), a comonoid \((D, \varepsilon, \delta)\) and a map \(\psi : T \star D \to R\) such that

\[
\begin{align*}
\text{Id} \times \varepsilon : I \times I & \to I \\
\eta^R & \downarrow \\
I \times D & \to T \star D \to R \\
\eta \times \text{Id} & \downarrow \psi \\
T \star D & \to R \\
(T \diamond T) \times (D \diamond D) & \to (T \star D) \diamond (T \star D) \\
\text{Id} \times \delta & \downarrow \\
(T \diamond T) \times D & \to (T \star D) \diamond D \\
\mu \times \text{Id} & \downarrow \psi \\
\mu \times \text{Id} & \downarrow \\
T \star D & \to R \\
\psi \diamond \psi & \downarrow \\
R \diamond R & \to R \\
\mu^R & \downarrow \\
R \diamond R & \to R
\end{align*}
\]