

Monads and Interaction

Lecture 2

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Monad algebras

Monad algebras

- An *algebra* of a monad (T, η, μ) is an object X with a map $\xi : TX \rightarrow X$ such that

$$\begin{array}{ccc} X & & T(TX) \xrightarrow{T\xi} TX \\ \eta_X \downarrow & \searrow & \downarrow \mu_X \quad \downarrow \xi \\ TX & \xrightarrow{\xi} & X \end{array}$$

- A *map* between two algebras (Y, χ) and (X, ξ) is a map h such that

$$\begin{array}{ccc} TY & \xrightarrow{Th} & TX \\ \chi \downarrow & & \downarrow \xi \\ Y & \xrightarrow{h} & X \end{array}$$

- The algebras of the monad and maps between them form a category $\mathbf{Alg}(T)$, called the *Eilenberg-Moore category*, with an obvious forgetful functor $U : \mathbf{Alg}(T) \rightarrow \mathcal{C}$.

Kleisli triple algebras

- A variation of algebras fitting more smoothly with Kleisli triples is this.
- A *algebra* of a Kleisli triple $(T, \eta, (-)^*)$ (a *Mendler-style algebra*, an *algebra in extension form*, *no-iteration form*) is given by
 - an object X ,
 - a family of maps $(-)_Y^+ : \mathcal{C}(Y, X) \rightarrow \mathcal{C}(TY, X)$ indexed by $Y \in |\mathcal{C}|$such that
 - if $f : Y \rightarrow X$, then $f^+ \circ \eta_Y = f$
 - if $k : Z \rightarrow TY$, $f : Y \rightarrow X$, then $(f^+ \circ k)^+ = f^+ \circ k^* : TZ \rightarrow X$
- Naturality of $(-)^+$ is not required, it follows.
- There's also the correct concept of Kleisli triple algebra map. (Definition omitted.)

Monad algebras = Kleisli triple algebras

- Algebras of monads/Kleisli triples with the same carrier X are in a bijection.
- This is again crucially by the Yoneda lemma.

$$\frac{TX \rightarrow X}{\mathcal{C}(Y, X) \rightarrow \mathcal{C}(TY, X) \text{ nat. in } Y}$$

- From ξ , one defines $(-)^+$ by $f^+ = \xi \circ Tf$.
 - From $(-)^+$, one defines ξ by $\xi = \text{id}_X^+$.
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- The respective categories are isomorphic.

FP intuition

- An algebra of a monad T with carrier X is a “handler” of computations of values of the type X (and only of that type!).
- $\xi : TX \rightarrow X$ –
a value of X can be extracted from a computation of values of X
- $(-)_Y^+ : \mathcal{C}(Y, X) \rightarrow \mathcal{C}(TY, X)$ –
given a way $f : Y \rightarrow X$ to “observe” values of Y as values of X ,
 $f^+ : TY \rightarrow X$ is a way of observing computations of values of Y

Eilenberg-Moore adjunction

- In the opposite direction of $U : \mathbf{Alg}(T) \rightarrow \mathcal{C}$ there is a functor $L : \mathcal{C} \rightarrow \mathbf{Alg}(T)$ defined by
 - $LX = (TX, \mu_X)$,
 - $Lf = Tf : (TY, \mu_Y) \rightarrow (TX, \mu_X)$ for $f : Y \rightarrow X$.
- L is left adjoint to U .

$$\begin{array}{ccc}
 \mathbf{Alg}(T) & & \overbrace{(TY, \mu_Y)}^{LY} \rightarrow (X, \xi) \\
 \uparrow L \quad \downarrow U & & \hline
 \mathcal{C} & & Y \rightarrow \underbrace{X}_{U(X, \xi)}
 \end{array}$$

- This says that (TX, μ_X) is an algebra of the monad T , moreover, it is the free one.
- $U \cdot L = T$. Indeed,
 - $U(LX) = U(TX, \mu_X) = TX$,
 - if $f : Y \rightarrow X$, then $U(Lf) = U(Tf) = Tf$.
- The unit of the adjunction is η .
- The E-M resolution of a monad is its final resolution.

Algebras of exceptions monads

- Algebras of the exceptions monad $TX = E + X$ are (by definition) objects X with a map $\xi : E + X \rightarrow X$ subject to 2 equations.
- They are in a bijection with pairs of an object X and map $E \rightarrow X$.
- The E-M category of this monad is isomorphic to the coslice category E/\mathcal{C} .

- [FP intuition] These are handlers for exceptional computations!
- To be able to extract a value from any given exceptional computation, you must know how to deal with the exception case.

Algebras of reader monads

- Algebras of the reader monad $TX = S \Rightarrow X$ are (by definition) objects X with a map $get : S \Rightarrow X \rightarrow X$ such that
 - $get(\lambda s. x) = x$
 - $get(\lambda s. get(\lambda s'. f s s')) = get(\lambda s. f s s)$

Algebras of state monads

- The E-M category of the state monad $TX = S \Rightarrow S \times X$ is isomorphic to the category of mnemoids.
- An algebra of this monad is an object X with a map $getput : S \Rightarrow S \times X \rightarrow X$ such that
 - $x = getput (\lambda s. (s, x))$
 - $getput (\lambda s. let (s', g) = f s \text{ in } (s', getput g)) = getput (\lambda s. let (s', g) = f s \text{ in } g s')$
- A *mnemoid* is an object X with maps $get : S \Rightarrow X \rightarrow X$ and $put : S \times X \rightarrow X$ such that
 - $x = get (\lambda s. put (s, x))$
 - $put (s, get f) = put (s, f s)$
 - $put (s, put (s', x)) = put (s', x)$
- From ξ , one constructs get , put by $get f = \xi (\lambda s. (s, f s))$,
 $put (s, x) = \xi (\lambda _. (s, x))$.
- From get , put , one obtains ξ by $\xi f = get (\lambda s. put (f s))$.

Algebras of list monads

- The E-M category of the standard list monad is isomorphic to that of monoids,
i.e., objects X with maps $1 \rightarrow X$ and $X \times X \rightarrow X$ satisfying left and right unitality and associativity.
- It is therefore also called the *free monoids monad*.

- The E-M category of the alternative list monad is in a bijection with semigroups with zero.
- A *semigroup with zero* is an object X with maps $1 \rightarrow X$ and $X \times X \rightarrow X$ satisfying left and right zeroness and associativity.

Algebras of free functor-algebras monads

- The E-M category $\mathbf{Alg}(F^*)$ of the monad F^* of free algebras of a functor F is isomorphic to the category $\mathbf{alg}(F)$ of algebras of F

$$\begin{array}{ccc} \mathbf{Alg}(F^*) & \xrightarrow{\cong} & \mathbf{alg}(F) \\ & \searrow U & \swarrow U \\ & \mathcal{C} & \end{array}$$

- A monad with this property is said to be *algebraically free* on F .
- For $FX = X \times X$, algebras with carrier X of the monad $T = F^*$ (the “intensional” nondeterminism monad) are maps $\mu: X + X \times X \rightarrow X$ subject to two equations.
- They are in bijection with algebras with carrier X of the functor F , which are maps $X \times X \rightarrow X$ subject to no conditions (*magmas*).

Algebras of free functor-algebras monads ctd

- For $FX = (S \Rightarrow X) + (S \times X)$, algebras with carrier X of the monad $T = F^*$ (the “intensional” state monad) are maps $\mu_Z : X + (S \Rightarrow Z) + (S \times Z) \rightarrow X$ subject to two equations.
- They are in bijection with algebras with carrier X of the functor F , which are pairs of maps $get : S \Rightarrow X \rightarrow X$ and $put : S \times X \rightarrow X$ subject to no conditions (we can call them *premonoids*).
- Algebras of the ordinary (“extensional”) state monad $TX = S \Rightarrow (S \times X)$ are a similar structure, but subject to the 3 equations of a monoid.

Monad maps

Monad maps

- A *monad map* between monads T, T' on a category \mathcal{C} is a natural transformation $\tau : T \rightarrow T'$ satisfying

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ \eta_X \downarrow & & \downarrow \eta'_X \\ TX & \xrightarrow{\tau_X} & T'X \end{array} \qquad \begin{array}{ccccc} T(TX) & \xrightarrow{\tau_{TX}} & T'(TX) & \xrightarrow{T'\tau_X} & T'(T'X) \\ \mu_X \downarrow & & & & \downarrow \mu'_X \\ TX & \xrightarrow{\tau_X} & & & T'X \end{array}$$

- Monads on \mathcal{C} and maps between them form a category **Monad**(\mathcal{C}).
- **Monad**(\mathcal{C}) is the category of monoids in the (strict) monoidal category $([\mathcal{C}, \mathcal{C}], \text{Id}_{\mathcal{C}}, \cdot)$.

Kleisli triple maps

- A map between two Kleisli triples T, T' is a family of maps $\tau_X : TX \rightarrow T'X$ indexed by $X \in |C|$ such that
 - $\tau_X \circ \eta_X = \eta'_X$,
 - if $k : X \rightarrow TY$, then $\tau_Y \circ k^* = (\tau_Y \circ k)^{*'} \circ \tau_X$.
- Naturality of τ is not required, but it follows.
- Kleisli triples on C and maps between them form a category isomorphic to **Monad**(C).

Maps between exceptions, reader, writer monads

- Monad maps between the exception monads for sets E, E' are in a bijection with pairs of maps $1 \rightarrow E' + 1$ and $E \rightarrow E'$.
- Monad maps between the reader monads for sets S, S' are in a bijection with maps between S', S .
- Monad maps between the writer monads for monoids (P, o, \oplus) and (P', o', \oplus') are in a bijection with homomorphisms between these monoids.

Maps between state monads

- The monad maps between the state monads for S and S_0 are in a bijection with (*very well-behaved*) lenses.
- These are pairs of maps $coget : S_0 \rightarrow S$, $coput : S_0 \times S \rightarrow S_0$ such that
 - $s_0 = coput (s_0, coget s_0)$,
 - $coget (coput (s_0, s)) = s$,
 - $coput (coput (s_0, s), s') = coput (s_0, s')$.

Free functor-algebras monads are free

- The monad F^* of free algebras of a functor F (the algebraically-free monad on F), if it exists, is the free monad on F .

$$\begin{array}{ccc} & \mathbf{Monad}(\mathcal{C}) & \\ & \left(\begin{array}{c} \uparrow \\ (-)^* \quad \dashv \\ \downarrow \end{array} \right) \cup & \frac{F^* \rightarrow T}{F \rightarrow UT} \\ & [\mathcal{C}, \mathcal{C}] & \end{array}$$

- (Use the full subcategory of $[\mathcal{C}, \mathcal{C}]$ of those functors for which $(-)^*$ exists.)
- If a monad T is free on F , it need not be algebraically-free on F .
- A monad T is free on F iff $T \cong \mu H. \text{Id} + F \cdot H$.
- It is algebraically free iff $TX \cong \mu Z. X + F(TX)$. This is generally a stronger condition.

Maps to continuation monads

- Let xCnt^R be the external continuation monad for R
($\text{xCnt}^R X = \mathcal{C}(X, R) \multimap R$).
- Monad maps between an arbitrary monad T and the monad xCnt^R are in a bijection with algebras of T with carrier R .
- Yoneda strikes again. :-)

$$\frac{\frac{TR \rightarrow R}{\mathcal{C}(X, R) \rightarrow \mathcal{C}(TX, R) \text{ nat. in } X}}{TX \rightarrow \mathcal{C}(X, R) \multimap R \text{ nat. in } X}}$$

- Let Cnt^R be the continuation monad for R , which is strong.
- Strong monad maps between an arbitrary strong monad T and Cnt^R are in a bijection with algebras T with carrier R .

Monad maps vs. functors between Kleisli categories

- There is a bijection between monad maps $\tau : T \rightarrow T'$ and functors $V : \mathbf{KI}(T) \rightarrow \mathbf{KI}(T')$ such that

$$\begin{array}{ccc} \mathbf{KI}(T) & \xrightarrow{V} & \mathbf{KI}(T') \\ & \swarrow J & \nearrow J' \\ & \mathcal{C} & \end{array}$$

- This is defined by
 - $VX = X$,
 - $Vk = Y \xrightarrow{k} TX \xrightarrow{\tau_X} T'X$ for $k : Y \rightarrow TX$and
- $\tau_X = V(TX \xrightarrow{\text{id}_{TX}} TX) : TX \rightarrow T'X$.

Monad maps vs. functors between E-M categories

- There is a bijection between monad maps $\tau : T \rightarrow T'$ and functors $V : \mathbf{Alg}(T') \rightarrow \mathbf{Alg}(T)$ such that

$$\begin{array}{ccc} \mathbf{Alg}(T') & \xrightarrow{V} & \mathbf{Alg}(T) \\ & \searrow u' & \swarrow u \\ & \mathcal{C} & \end{array}$$

(Note the reversed direction.)

- This is defined by
 - $V(X, \xi) = (X, TX \xrightarrow{\tau_X} T'X \xrightarrow{\xi} X)$,
 - $Vh = h : (Y, \chi \circ \tau_Y) \rightarrow (X, \xi \circ \tau_X)$ for $h : (Y, \chi) \rightarrow (X, \xi)$
- and
- $\tau_X = \text{let } (T'X, \zeta) = V(T'X, \mu'_X) \text{ in } \zeta \circ T\eta'_X.$