

# Monads and Interaction

## Lecture 3

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# Comonads



# Comonads

- A *comonad* on a category  $\mathcal{C}$  is given by a
  - a functor  $D : \mathcal{C} \rightarrow \mathcal{C}$ ,
  - a natural transformation  $\varepsilon : D \rightarrow \text{Id}_{\mathcal{C}}$  (the *counit*),
  - a natural transformation  $\delta : D \rightarrow D \cdot D$  (the *comultiplication*)

such that

$$\begin{array}{ccc} DY & \xrightarrow{\delta_Y} & D(DY) \\ \delta_Y \downarrow & \searrow & \downarrow D\varepsilon_Y \\ D(DY) & \xrightarrow{\varepsilon_{DY}} & DY \end{array} \qquad \begin{array}{ccc} DY & \xrightarrow{\delta_Y} & D(DY) \\ \delta_{DY} \downarrow & & \downarrow D\delta_Y \\ D(DY) & \xrightarrow{\delta_{DY}} & D(D(DY)) \end{array}$$

## FP intuition

- $D$  – a “notion” of environment (environments of some flavor with whose help computations can run)
- An environment  $\approx$  a process that is able to serve computations and is at any moment in some state.
- $Y$  – states
- $DY$  – environments with states of type  $Y$
- $D(DY)$  – environments with, as states, environments with states of type  $Y$
  
- $\varepsilon_Y : DY \rightarrow Y$  – extract from an environment its initial state (useful for “halting” the environment at the start moment)
- $\delta_Y : DY \rightarrow D(DY)$  – “blow up” an environment into one that has as its state at any given moment the current remainder of the environment (useful for “pausing” the environment whenever, with the intent to “resume”)

# CoKleisli triples

- A *coKleisli triple* (comonad in extension form, no-iteration form) is given by
  - an object mapping  $D : |\mathcal{C}| \rightarrow |\mathcal{C}|$ ,
  - a family of maps  $\varepsilon_Y : DY \rightarrow Y$  indexed by  $X \in |\mathcal{C}|$ ,
  - a family of maps  $(-)^{\dagger}_{X,Y} : \mathcal{C}(DX, Y) \rightarrow \mathcal{C}(DX, DY)$  indexed by  $X, Y \in |\mathcal{C}|$  (the *coKleisli coextension* operation)

such that

- $\varepsilon_Y \circ k^{\dagger} = k$  for  $k : DX \rightarrow Y$ ,
  - $\varepsilon_Y^* = \text{id}_{DY}$ ,
  - $(\ell \circ k^{\dagger})^{\dagger} = \ell^{\dagger} \circ k^{\dagger} : DX \rightarrow DZ$  for  $k : DX \rightarrow Y, \ell : DY \rightarrow Z$
- 
- Functoriality of  $D$ , naturality of  $\varepsilon, (-)^{\dagger}$  are not required, but follow.
- 
- Comonads = coKleisli triples, for the same  $D : |\mathcal{C}| \rightarrow |\mathcal{C}|, \varepsilon$ .

# CoKleisli category of a comonad

- A comonad  $D$  on a category  $\mathcal{C}$  induces a category **CoKl**( $D$ ) called the *coKleisli category* of  $D$  defined by

- an object is an object of  $\mathcal{C}$ ,
- a map of from  $X$  to  $Y$  is a map of  $\mathcal{C}$  from  $DX$  to  $Y$ ,
- ${}^D\text{id}_Y = DY \xrightarrow{\varepsilon_Y} Y$ ,

- $l \circ k = DX \xrightarrow{\delta_X} DDX \xrightarrow{Dk} DY \xrightarrow{\ell} Z$

for  $k : X^D \rightarrow Y$ ,  $\ell : Y^D \rightarrow Z$

- From  $\mathcal{C}$  there is an identity-on-objects functor  $J$  to **CoKl**( $T$ ), defined on maps by

- $Jf = DX \xrightarrow{\varepsilon_X} X \xrightarrow{f} Y$  for  $f : X \rightarrow Y$

- If  $\varepsilon$  is epi, then  $J$  is faithful.





# Comonad coalgebras

- A *coalgebra* of a comonad  $(D, \varepsilon, \delta)$  is an object  $Y$  with a map  $\chi: Y \rightarrow DY$  such that

$$\begin{array}{ccc} Y & & Y \xrightarrow{\chi} DY \\ \chi \downarrow & \searrow & \downarrow D\chi \\ DY & \xrightarrow{\varepsilon_Y} & Y \\ & & DY \xrightarrow{\delta_Y} D(DY) \end{array}$$

- A *map* between two coalgebras  $(X, \xi)$  and  $(Y, \chi)$  is a map  $h$  such that

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ \xi \downarrow & & \downarrow \chi \\ DX & \xrightarrow{Th} & DY \end{array}$$

- The coalgebras of the comonad and maps between them form a category  $\mathbf{Coalg}(D)$ , called the *coEilenberg-Moore category*, with an obvious forgetful functor  $U: \mathbf{Coalg}(T) \rightarrow \mathcal{C}$ .

## FP intuition

- A coalgebra  $(Y, \xi : Y \rightarrow DY)$  – a coeffect producer (should we call it a cohandler?)
- $Y$  – a fixed type of states
- $\xi : Y \rightarrow DY$  –  
a (consistent) assignment of an environment to every initial state

(think, e.g., that you have a state machine over state set  $Y$  that you can start in any state and obtain a behavior)

# CoEilenberg-Moore adjunction

- In the opposite direction of  $U : \mathbf{Coalg}(D) \rightarrow \mathcal{C}$  there is a functor  $R : \mathcal{C} \rightarrow \mathbf{Coalg}(D)$  defined by
  - $RY = (DY, \delta_Y)$ ,
  - $Rf = Df : (DX, \delta_X) \rightarrow (DY, \delta_Y)$  for  $f : X \rightarrow Y$ .
- $R$  is right adjoint to  $U$ .

$$\begin{array}{ccc}
 \mathbf{Coalg}(D) & & \\
 U \swarrow \quad \nearrow R & & \\
 \mathcal{C} & & 
 \end{array}
 \quad
 \frac{(X, \xi) \rightarrow \overbrace{(DY, \delta_Y)}^{RY}}{\underbrace{X}_{U(X, \xi)} \rightarrow Y}$$

- This says that  $(DY, \delta_X)$  is a coalgebra of the comonad  $D$ , moreover, it is the cofree one.
- $U \cdot R = D$ . Indeed,
  - $U(RY) = U(DY, \delta_Y) = DY$ ,
  - if  $f : X \rightarrow Y$ , then  $U(Rf) = U(Df) = Df$ .
- The counit of the adjunction is  $\varepsilon$ .

# Comonads from adjunctions

- Any adjunction gives rise to a comonad.
- Given an adjunction

$$\begin{array}{ccc} & \mathcal{D} & \\ L \swarrow & \dashv & \searrow R \\ & \mathcal{C} & \end{array} \quad \frac{LX \rightarrow Y}{X \rightarrow RY}$$

the endofunctor  $D = L \cdot R$  on  $\mathcal{C}$  carries a comonad structure with  $\varepsilon$  the counit of the adjunction.

- Adjunctions so related to a comonad are called its resolutions.
- Resolutions of a comonad form a category.
- The coKleisli and coE-M adjunctions are resolutions. The coKleisli resolution is the initial, the coE-M resolution the final object of this category.

# Coreader and cowriter comonads

- The *coreader* comonad for an object  $S$  (of states) is
  - $DY = S \times Y$
  - $\varepsilon_Y : S \times Y \rightarrow Y$   
 $\varepsilon(-, y) = y$
  - $\delta_Y : S \times Y \rightarrow S \times (S \times Y)$   
 $\delta(s, y) = (s, (s, y))$
  
- The *cowriter* comonad for a monoid  $(P, o, \oplus)$  (of updates) is
  - $DY = P \Rightarrow Y$
  - $\varepsilon_Y : P \Rightarrow Y \rightarrow Y$   
 $\varepsilon e = e o$
  - $\delta_Y : P \Rightarrow Y \rightarrow P \Rightarrow P \Rightarrow Y$   
 $\delta e = \lambda p. \lambda p'. e (p \oplus p')$

# Costate comonads

- The *costate comonad* (originally called *array comonad*) for an object  $S$  (of states) is
  - $DY = S \times (S \Rightarrow Y)$
  - $\varepsilon_Y : S \times (S \Rightarrow Y) \rightarrow Y$   
 $\varepsilon(s, f) = f s$
  - $\delta_Y : S \times (S \Rightarrow Y) \rightarrow S \times (S \Rightarrow S \times (S \Rightarrow Y))$   
 $\delta(s, f) = (s, \lambda s'. (s', f))$
- A variation, the *external costate comonad* for an object  $S$  of states, is
  - $DY = \mathcal{C}(S, Y) \bullet S$
- The costate monads for  $S$  arise from the adjunction

$$- \times S \left( \begin{array}{c} \mathcal{C} \\ \dashv \\ \mathcal{C} \end{array} \right) S \Rightarrow - \quad \frac{X \times S \rightarrow Y}{X \rightarrow S \Rightarrow Y}$$

$$- \bullet S \left( \begin{array}{c} \mathcal{C} \\ \dashv \\ \mathbf{Set} \end{array} \right) \mathcal{C}(S, -) \quad \frac{I \bullet S \rightarrow Y}{I \rightarrow \mathcal{C}(S, Y)}$$

# Coalgebras of costate comonads

- Coalgebras of the costate comonad for  $S$  are (by definition) objects  $Y$  with a map  $\chi : Y \rightarrow S \times (S \Rightarrow Y)$  subject to two equations.
- They are in a bijection very well-behaved lenses between  $Y$  and  $S$ .

Those are (by definition) pairs of maps  $coget : Y \rightarrow S$ ,  
 $coput : Y \times S \rightarrow Y$  such that

- $y = coput(y, coget y)$ ,
- $coget(coput(y, s)) = s$ ,
- $coput(coput(y, s), s') = coput(y, s')$ .

# Cofree coalgebras comonad of a functor

- This comonad  $G^\dagger$  delivers (carriers of) cofree algebras of a functor  $G$ :

- $DY = \nu W. Y \times GW$

$$DY \cong^\nu Y \times G(DY)$$

- (nonwellfounded(!)  $G$ -branching trees  
with  $Y$ -labelled branching nodes)

- $\varepsilon t = \text{fst}(\text{out } t)$

- (extracts the root label of a tree)

- $\delta t = \text{out}^{-1}(t, G \delta(\text{snd out } t))$

- (relabels each node of a tree with  
the subtree rooted by that node)

- For  $GY = Y$ , we get  $DY = \nu W. Y \times W \cong \text{Str } Y$   
(streams and suffixes).

- For  $GY = 1 + Y$ , we get  $DY = \nu W. Y \times (1 + W) \cong \text{NEColist } Y$   
(nonempty colists and suffixes).



## Cofree functor-coalgebras comonads

- For  $GY = (S \times Y) \times (S \Rightarrow Y)$ , we get  
 $DY = \nu W. Y \times (S \times W) \times (S \Rightarrow W)$   
(an “intensional” costate comonad for  $S$ )
- The ordinary (“extensional”) costate comonad  
 $DY = S \times (S \Rightarrow Y)$   
is a subcomonad obtained by only keeping those trees that are “good” in a certain sense.

# Coalgebras of cofree functor-coalgebras comonads

- The category  $\mathbf{Coalg}(G^\dagger)$  of coalgebras of the comonad  $G^\dagger$  is isomorphic to the category  $\mathbf{coalg}(G)$  of coalgebras of the functor  $G$ . The isomorphism is identity on carriers.
- E.g., the a coalgebra of the stream comonad is an object  $Y$  with a map  $\chi : Y \rightarrow \text{Str } Y$  subject to two equations.
- It is described by map  $\psi : Y \rightarrow Y$  subject to no equations.
- $\chi$  is defined from  $\psi$  by  $\text{shd}(\chi y) = y$ ,  $\text{stl}(\chi y) = \chi(\psi y)$
- $\psi$  is constructed from  $\chi$  by  $\psi y = \text{shd}(\text{stl}(\chi y))$ .

## Coalgebras of cofree functor-coalgebras comonad ctd

- A coalgebra of the “intensional” costate comonad is an object  $Y$  with a map  $\chi : Y \rightarrow \nu W. Y \times (S \times W) \times (S \Rightarrow W)$  subject to two equations.
  - It is described by maps  $coget' : Y \rightarrow S \times Y$  and  $coput : Y \rightarrow S \Rightarrow Y$  subject to no equations.
  - In contrast, a coalgebra of the standard (“extensional”) costate comonad is an object  $Y$  with maps  $coget : Y \rightarrow S$  and  $coput : Y \rightarrow S \Rightarrow Y$  subject to the lens equations.
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- The comonad  $G^\dagger$  is the free comonad on the functor  $G$ .