# Monads and interaction: Lecture 4

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OPLSS 2021, Eugene, OR, 14–26 June 2021

Monad-comonad interaction laws

# Effects happen in interaction

- To run,

  an effectful (effect-requesting) program behaving as
  a computation

  needs to **interact** with

  a environment
  that an effect-providing (coeffectful) machine behaves as

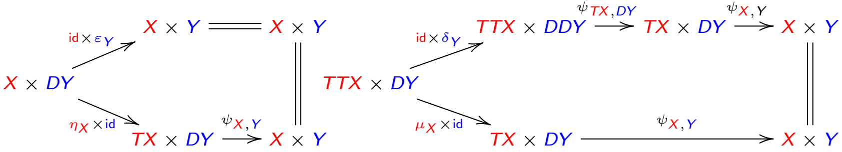
- E.g.,
  - a nondeterministic program needs a machine making choices;
  - a stateful program needs a machine coherently responding to fetch and store commands.

# Monad-comonad interaction laws

- Let $\mathcal{C}$ be a Cartesian category. (Symmetric monoidal works too.)
- A *monad-comonad interaction law* is given by a monad $(T, \eta, \mu)$ and a comonad $(D, \varepsilon, \delta)$ and a nat. transf. $\psi$ typed

$$\psi_{X,Y} : TX \times DY \to X \times Y$$

such that




- Legend:
  $X$ – values, $TX$ – computations
  $Y$ – states, $DY$ – environments (incl an initial state)

# Reader monads

- $TX = S \Rightarrow X$ (the reader monad),
  $DY = S_0 \times Y$ (the coreader comonad)
  for some $S_0$, $S$ and $c : S_0 \to S$

- $\psi\,(f, (s_0, y)) = (f\,(c\,s_0), y)$

- Legend:
  $X$ – values, $S$ – "views" of stores (data states),
  $Y$ – (control) states, $S_0$ – stores (data states)

# State monads

- $TX = S \Rightarrow (S \times X)$ (the state monad),
  $DY = S_0 \times (S_0 \Rightarrow Y)$ (the costate comonad)
  for some $S_0$, $S$, $c : S_0 \to S$ and $d : S_0 \times S \to S_0$
  forming a *(very well-behaved) lens*

- $\psi\,(f,(s_0,g)) = \text{let }(s',x) = f\,(c\,s_0)\text{ in }(x, g\,(d\,(s_0,s')))$

- Legend:
  $X$ – values, $S$ – "views" of stores (data states),
  $Y$ – (control) states, $S_0$ – stores (data states)

# Free functor-algebras monads (free monads)

- Intensional nondeterminism monad:
- $TX = \mu Z.\, X + Z \times Z$,

  $DY = \nu W.\, Y \times (W + W)$

  $\psi_{X,Y} : TX \times DY \to X \times Y$

  $\psi\,(\mathsf{in}\,(\mathsf{inl}\,x), e) = (x, \mathsf{fst}\,(\mathsf{out}\,e))$

  $\psi\,(\mathsf{in}\,(\mathsf{inr}\,(c_0, c_1)), e) = \mathsf{case}\ \mathsf{snd}\,(\mathsf{out}\,e)\ \mathsf{of}\ \begin{cases} \mathsf{inl}\ e' \mapsto \psi\,(c_0, e') \\ \mathsf{inr}\ e' \mapsto \psi\,(c_1, e') \end{cases}$

- Intensional state monads:
- $TX = \mu Z.\, X + (S \Rightarrow Z) + (S \times Z)$,

  $DY = \nu W.\, Y \times (S \times W) \times (S \Rightarrow W)$

  $\psi_{X,Y} : TX \times DY \to X \times Y$

  $\psi\,(\mathsf{in}\,(\mathsf{inl}\,x), e) = (x, \mathsf{fst}\,(\mathsf{out}\,e))$

  $\psi\,(\mathsf{in}\,(\mathsf{inr}\,(\mathsf{inl}\,f)), e) = \mathsf{let}\ (s, e') = \mathsf{fst}\,(\mathsf{snd}\,(\mathsf{out}\,e))\ \mathsf{in}\ \psi\,(f\,s, e')$
  $\psi\,(\mathsf{in}\,(\mathsf{inr}\,(\mathsf{inr}\,(s, c))), e) = \psi\,(c, \mathsf{snd}\,(\mathsf{snd}\,(\mathsf{out}\,e))\,s)$

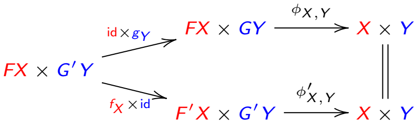
# Monad-comonad interaction laws are monoids

- A *functor-functor interaction law* is given by two functors $F, G : \mathcal{C} \to \mathcal{C}$ and a family of maps

$$\phi_{X,Y} : FX \times GY \to X \times Y$$

natural in $X, Y$.

- A *functor-functor interaction law map* between $(F, G, \phi)$, $(F', G', \phi')$ is given by nat. transfs. $f : F \to F'$, $g : G' \to G$ such that




- Functor-functor interaction laws form a category with a composition-based monoidal structure.

- These categories are isomorphic:
  - monad-comonad interaction laws;
  - monoid objects of the category of functor-functor interaction laws.

# Some degeneracy thms for func-func int laws

- Assume $\mathcal{C}$ is extensive ("has well-behaved coproducts").
- If $F$ has a nullary operation, i.e., a family of maps

  $$c_X : 1 \to FX$$

  natural in $X$ (eg, $F = \text{Maybe}$)
  or a binary commutative operation, i.e., a family of maps

  $$c_X : X \times X \to FX$$

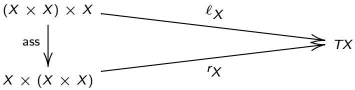
  natural in $X$ such that

  $$\begin{array}{ccc} X \times X & \xrightarrow{c_X} & \\ {\scriptstyle \text{sym}}\downarrow & & FX \\ X \times X & \xrightarrow{c_X} & \end{array}$$

  (eg, $F = \mathcal{M}^{+}_{\text{fin}}$) and $F$ interacts with $G$, then $GY \cong 0$.

# A degeneracy thm for mnd-cmnd int laws

- If $T$ has a binary associative operation, ie a family of maps $c_X : X \times X \to TX$ natural in $X$ such that

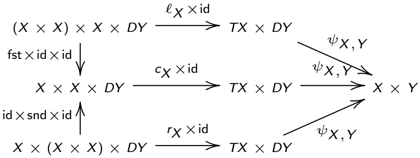



where

$$\ell_X = (X \times X) \times X \xrightarrow{c_X \times \eta_X} TX \times TX \xrightarrow{c_{TX}} TTX \xrightarrow{\mu_X} TX$$

$$r_X = X \times (X \times X) \xrightarrow{\eta_X \times c_X} TX \times TX \xrightarrow{c_{TX}} TTX \xrightarrow{\mu_X} TX$$

(eg, $T = \mathrm{List}^+$), then any int law $\psi$ of $T$ and $D$ obeys



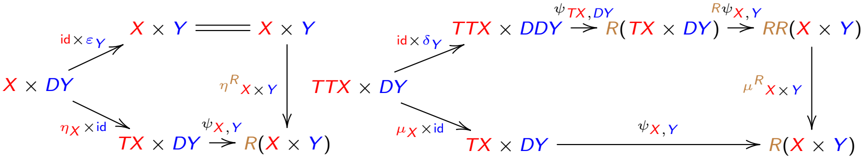
# Residual interaction laws

- Given a monad $(R, \eta^R, \mu^R)$ on $\mathcal{C}$.
- Eg, $R = \mathrm{Maybe}$, $\mathcal{M}^+_{\mathrm{fin}}$ or $\mathcal{M}_{\mathrm{fin}}$.

- A *residual monad-comonad interaction law* is given by a monad $(T, \eta, \mu)$, a comonad $(D, \varepsilon, \delta)$ and a family of maps

$$\psi_{X,Y} : TX \times DY \to R(X \times Y)$$

natural in $X$, $Y$ such that



# Residual interaction laws ctd

- A *residual functor-functor interaction law* is given by two functors $F, G : \mathcal{C} \to \mathcal{C}$ and a family of maps

$$\phi_{X,Y} : FX \times GY \to R(X \times Y)$$

natural in $X$, $Y$.

- $R$-residual functor-functor interaction laws form a monoidal category with $R$-residual monad-comonad interaction laws as monoids.

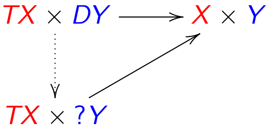
# Intensional nondeterminism (incl nullary choice)

- $TX = \mu Z.\, X + (1 + Z \times Z)$
  (nullary-binary leaf trees)
- $DY = \nu W.\, Y \times (W + W)$
- $RZ = \text{Maybe} Z = 1 + Z$

# Duals

# Duals

- Given a functor/monad/comonad, is there a "greatest" functor/comonad/monad interacting with it?




- The same question makes sense in the presense of a residual monad $R$.

# Dual of a functor

- Assume again that $\mathcal{C}$ is Cartesian closed
  (or symm monoidal closed).
- For a functor $G : \mathcal{C} \to \mathcal{C}$, its *dual* is the functor $G^\circ : \mathcal{C} \to \mathcal{C}$ is

  $$G^\circ X = \int_Y GY \Rightarrow (X \times Y)$$

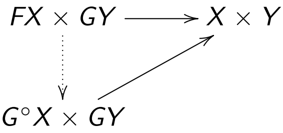
  (if this end exists).
- $(-)^\circ$ is a functor $[\mathcal{C}, \mathcal{C}]^{\mathrm{op}} \to [\mathcal{C}, \mathcal{C}]$
  (if all functors $\mathcal{C} \to \mathcal{C}$ are dualizable;
  if not, restrict to some full subcategory of $[\mathcal{C}, \mathcal{C}]$ closed under dualization).
- $G^\circ = G \multimap_\star \mathrm{Id}$ where $G \multimap_\star (-)$ is the right adjoint of $(-) \star G$ and $F \star G$ is the Day convolution of $F$ and $G$.

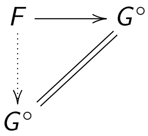
# Dual of a functor ctd

- The dual $G^\circ$ is the "greatest" functor interacting with $G$.

- These categories are isomorphic:
  - functor-functor interaction laws;
  - pairs of functors $F$, $G$ with nat. transfs. $F \to G^\circ$;
  - pairs of functors $F$, $G$ with nat. transfs. $G \to F^\circ$.

$$\frac{FX \times GY \to X \times Y \text{ nat in } X, Y}{FX \to \underbrace{\int_Y GY \Rightarrow (X \times Y)}_{G^\circ X} \text{ nat in } X}$$



# Some examples of dual

- For $GY = 0$, we have $G^\circ X \cong 1$
  and, for $GY = G_0 Y + G_1 Y$, we have $G^\circ X \cong G_0^\circ X \times G_1^\circ X$.

- For $GY = 1$, we have $G^\circ X \cong 0$.
- For $GY = A \times G' Y$, we have $G^\circ X \cong A \Rightarrow G'^\circ X$.

- For $GY = A \Rightarrow Y$, we have $G^\circ X \cong A \times X$.
- For $GY = A \Rightarrow G' Y$, we only have $G^\circ X \leftarrow A \times G'^\circ X$.

- $\mathrm{Id}^\circ \cong \mathrm{Id}$.
- But we only have $(G_0 \cdot G_1)^\circ \leftarrow G_0^\circ \cdot G_1^\circ$.
- For any $G$ with a nullary or a binary commutative operation, we have $G^\circ X \cong 0$.

# Dual of a comonad / Sweedler dual a monad

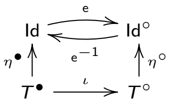
- The dual $D^\circ$ of a comonad $D$ is a monad.
- This is because $(-)^\circ : [\mathcal{C}, \mathcal{C}]^{\mathrm{op}} \to [\mathcal{C}, \mathcal{C}]$ is lax monoidal, so send monoids to monoids.

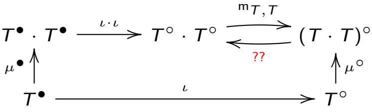
- But $(-)^\circ$ is not oplax monoidal, does not send comonoids to comonoids.
- So the dual $T^\circ$ of a monad $T$ is generally not a comonad.

- However we can talk about the *Sweedler dual* $T^\bullet$ of $T$.
- Informally, it is defined as the greatest functor $D$ that is smaller than the functor $T^\circ$ and carries a comonad structure $\eta^\bullet$, $\mu^\bullet$ agreeing with $\eta^\circ$, $\mu^\circ$.

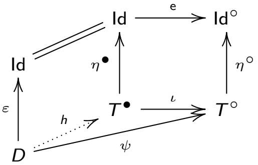
# Dual of a comonad / Sweedler dual of a monad ctd

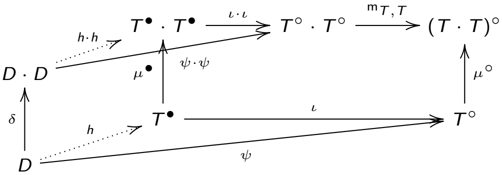
- Formally, the *Sweedler dual* of the monad $T$ is the comonad $(T^\bullet, \eta^\bullet, \mu^\bullet)$ together with a natural transformation $\iota : T^\bullet \to T^\circ$ such that




and such that, for any comonad $(D, \varepsilon, \delta)$ together with a natural transformation $\psi$ satisfying the same conditions, there is a unique comonad map $h : D \to T^\bullet$ satisfying



# Some examples of dual and Sweedler dual

- Let $TX = \mathrm{List}^+ X \cong \Sigma n : \mathbb{N}. ([0..n] \Rightarrow X)$ (the nonempty list monad) .
- We have $T^\circ Y \cong \Pi n : \mathbb{N}. ([0..n] \times Y)$ but $T^\bullet Y \cong Y \times (Y + Y)$.
- Let $TX = S \Rightarrow (S \times X) \cong (S \Rightarrow S) \times (S \Rightarrow X)$ (the state monad).
- We have $T^\circ Y = (S \Rightarrow S) \Rightarrow (S \times Y)$ but $T^\bullet Y = S \times (S \Rightarrow Y)$.

# An algebraic-coalgebraic perspective

# Stafeful runners

- Given
  - a resid mnd-cmnd int law, i.e., nat transf typed $\psi_{X,Y} : TX \times DY \to R(X \times Y)$ subject to eqns
  - a coEM coalgebra $(Y, \chi : Y \to DY)$ of $D$ (a "cohandler")

  we get
  - a nat transf typed $\theta_X : TX \times Y \to R(X \times Y)$ subject to other eqns (a *resid stateful runner*)

  by

$$\theta_X = \ TX \times Y \xrightarrow{TX \times \chi} TX \times DY \xrightarrow{\psi_{X,Y}} R(X \times Y)$$

- Where do these constructions with EM (co)algebras come from?

# Alternative definitions

- If $\mathcal{C}$ is Cartesian closed (or symmetric monoidal closed), $R$-resid mnd-cmnd int laws of $T, D$ can be defined in multiple ways:

$$\frac{\dfrac{TX \times DY \to R(X \times Y) \text{ nat in } X, Y \text{ subj to eqs}}{\mathcal{C}(X \times Y, Z) \to \mathcal{C}(TX \times DY, RZ) \text{ nat in } X, Y, Z \text{ subj to eqs}}}{\dfrac{T(Y \Rightarrow Z) \to DY \Rightarrow RZ \text{ nat in } Y, Z \text{ subj to eqs}}{D(X \Rightarrow Z) \to TX \Rightarrow RZ \text{ nat in } X, Z \text{ subj to eqs}}}$$

  (Yoneda again!)

  (A symm monoidal closed category will also do.)

- Legend:
  $X$ – values
  $Y$ – states
  $Z$ – observables
  (values for residual computations)
  $X \times Y \to Z$ – observation functions

# A (co)algebraic view

- Resid mnd-cmnd int laws are in a bijection with coalgebra-algebra exponentiation functors:

$$T(Y \Rightarrow Z) \to DY \Rightarrow RZ \text{ nat in } Y, Z \text{ subj to eqs}$$

$$\overline{\overline{\qquad\qquad\qquad\qquad\qquad\qquad}}$$

$$\begin{array}{ccc}
(\mathbf{coEM}(D))^{\mathrm{op}} \times \mathbf{EM}(R) & \longrightarrow & \mathbf{EM}(T) \\
\downarrow {\scriptstyle U^{\mathrm{op}} \times U} & & \downarrow {\scriptstyle U} \\
\mathcal{C}^{\mathrm{op}} \times \mathcal{C} & \xrightarrow{\Rightarrow} & \mathcal{C}
\end{array}$$

$$(Y, \chi : Y \to DY), (Z, \zeta : RZ \to Z) \mapsto (Y \Rightarrow Z, T(Y \Rightarrow Z) \to (Y \Rightarrow Z))$$

$$\overline{\overline{\qquad\qquad\qquad\qquad\qquad\qquad}}$$

$$\begin{array}{ccc}
(\mathbf{coKl}(D))^{\mathrm{op}} \times \mathbf{Kl}(R) & \longrightarrow & \mathbf{EM}(T) \\
\downarrow {\scriptstyle L^{D\,\mathrm{op}} \times R^{T}} & & \downarrow {\scriptstyle U} \\
\mathcal{C}^{\mathrm{op}} \times \mathcal{C} & \xrightarrow{\Rightarrow} & \mathcal{C}
\end{array}$$

# A (co)algebraic view ctd

- Explicitly, given a resid mnd-cmnd int law $\psi$,

  the corresponding (co)alg exp functor $E$ sends an EM-coalgebra $(Y, \chi)$ of $D$ and an EM-algebra $(Z, \zeta)$ of $R$ to the EM-algebra $(Y \Rightarrow Z, \xi)$ of $T$ where

$$\xi = \; T(Y \Rightarrow Z) \xrightarrow{\psi_{Y,Z}} DY \Rightarrow RZ \xrightarrow{\chi \Rightarrow \zeta} Y \Rightarrow Z$$

- Conversely, given a (co)alg exp functor $E$,

  the corresponding resid mnd-cmnd int law is

$$\psi_{Y,Z} = \; T(Y \Rightarrow Z) \xrightarrow{T(\varepsilon_Y \Rightarrow \eta^R_Z)} T(DY \Rightarrow RZ) \xrightarrow{e_{Y,Z}} DY \Rightarrow RZ$$

  where $(DY \Rightarrow RZ, e_{Y,Z}) = E((DY, \delta_Y), (RZ, \mu^R_Z))$.

# Intermediate views

- In fact the picture is finer, there are also two intermediate bijections:

$$
\begin{array}{ccc}
& \mathbf{MCIL}_R(T,D) & \\
\cong\ / & & \backslash\ \cong \\
[(\mathbf{coEM}(D))^{\mathrm{op}}, (\mathbf{SRun}_R(T))^{\mathrm{op}}]_{\mathrm{cp.}} & & [\mathbf{EM}(R), \mathbf{CRun}_D(T)]_{\mathrm{cp.}} \\
\cong\ \backslash & & /\ \cong \\
& [(\mathbf{coEM}(D)^{\mathrm{op}} \times \mathbf{EM}(R), \mathbf{EM}(T)]_{\mathrm{ce.}} &
\end{array}
$$

where

$\mathbf{SRun}_R(T)$ - $R$-residual stateful runners of $T$
$\mathbf{CRun}_D(T)$ - $D$-fuelled continuation-based runners of $T$

# Stateful runners

- For any $Y$, we have

*R-residual stateful runners* of $T$ w/ carrier $Y$, ie
$TX \times Y \to R(X \times Y)$ nat in $X$ subj to eqs

$$=\!=\!=\!=\!=\!=\!=\!=\!=\!=$$

monad morphisms from $T$ to $\mathrm{St}_Y^R$, ie
$TX \to Y \Rightarrow R(X \times Y)$ nat in $X$ subj to eqs

$$=\!=\!=\!=\!=\!=\!=\!=\!=\!=$$

$$\begin{array}{ccc} \mathbf{EM}(R) & \longrightarrow & \mathbf{EM}(T) \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathcal{C} & \xrightarrow{Y \Rightarrow (-)} & \mathcal{C} \end{array}$$

where $\mathrm{St}_Y^R$ is the *R-transformed state monad* for state object $Y$ given by

$$\mathrm{St}_Y^R X = Y \Rightarrow R(X \times Y)$$

# Continuation-based runners

- For any $Z$, we have

*$D$-fuelled continuation-based runners* of $T$ w/ carrier $Z$, ie
$D(X \Rightarrow Z) \to TX \Rightarrow Z$ nat in $X$ subj to eqs

$$=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=$$

monad morphisms from $T$ to $\mathsf{Cnt}^D_Z$, ie
$TX \to D(X \Rightarrow Z) \Rightarrow Z$ nat in $X$ subj to eqs

$$=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=$$

$$\begin{array}{ccc}
(\mathbf{coEM}(D))^{\mathrm{op}} & \longrightarrow & \mathbf{EM}(T) \\
{\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\
\mathcal{C} & \xrightarrow{(-)\Rightarrow Z} & \mathcal{C}
\end{array}$$

where $\mathsf{Cnt}^D_Z$ is the *$D$-transformed continuation monad* for answer object $Z$ given by

$$\mathsf{Cnt}^D_Z X = D(X \Rightarrow Z) \Rightarrow Z$$

# EM algebras of $T$ w/ carrier $Y \Rightarrow Z$ as runners

- For any $Y, Z$, we have

$$\frac{\dfrac{\begin{array}{c}\text{state and continuation based runners of } T \text{ w/ carrier } Z\text{, ie} \\ \mathcal{C}(X \times Y, Z) \to \mathcal{C}(TX \times Y, Z) \text{ nat in } X \text{ subj to eqs}\end{array}}{\begin{array}{l}\text{monad morphisms from } T \text{ to } \mathrm{xCntSt}_{Y,Z} \cong \mathrm{xCostCnt}_{Y,Z}\text{, ie} \\ \quad TX \to Y \Rightarrow \mathrm{xCnt}_Z(X \times Y) \\ \qquad\qquad \cong \ \mathrm{xCost}_Y(X \Rightarrow Z) \Rightarrow Z \text{ nat in } X \text{ subj to eqs}\end{array}}}{\text{EM algebras of } T \text{ with carrier } Y \Rightarrow Z}$$

where

$$\begin{array}{rcl}
\mathrm{xCnt}_Z X & = & \mathcal{C}(X, Z) \pitchfork Z \\
\mathrm{xCntSt}_{Y,Z} X & = & Y \Rightarrow \mathrm{xCnt}_Z(X \times Y) \\
& = & Y \Rightarrow (\mathcal{C}(X \times Y, Z) \pitchfork Z) \\
\mathrm{xCost}_Y X & = & \mathcal{C}(Y, X) \bullet Y \\
\mathrm{xCostCnt}_{Y,Z} X & = & \mathrm{xCost}_Y(X \Rightarrow Z) \Rightarrow Z \\
& = & (\mathcal{C}(Y, X \Rightarrow Z) \bullet Y) \Rightarrow Z
\end{array}$$

# Monoid-comonoid interaction laws

# Residual interaction laws and Chu spaces

- The *Day convolution* of $F$, $G$ is

$$(F \star G)Z = \int^{X,Y} \mathcal{C}(X \times Y, Z) \bullet (FX \times GY)$$

(if this coend exists).

- These categories are isomorphic for a given monad $R$:
  - $R$-residual functor-functor interaction laws;
  - Chu spaces on the symm monoidal category $([\mathcal{C},\mathcal{C}], J, \star)$ with vertex $R$, ie, triples of two functors $F$, $G$ with a nat transf $F \star G \to R$.

(if $\star$ is defined for all functors).

$$\frac{\dfrac{FX \times GY \to R(X \times Y) \text{ nat in } X, Y}{\mathcal{C}(X \times Y, Z) \to \mathcal{C}(FX \times GY, RZ) \text{ nat in } X, Y, Z}}{\underbrace{\int^{X,Y} \mathcal{C}(X \times Y, Z) \bullet (FX \times GY)}_{(F \star G)Z} \to RZ \text{ nat in } Z}$$

# Residual interaction laws and Chu spaces ctd

- We do not immediately get another chacterization of the category of $R$-residual monad-comonad interaction laws.
- We have to use that $[\mathcal{C}, \mathcal{C}]$ has a *duoidal* structure $(\mathsf{Id}, \cdot, J, \star)$.
- In particular, $\star$ is oplax monoidal wrt $(\mathsf{Id}, \cdot)$, so there are structural laws

$$\mathsf{Id} \star \mathsf{Id} \to \mathsf{Id}$$
$$(F \cdot F') \star (G \cdot G') \to (F \star G) \cdot (F' \star G')$$

  with the requisite properties.
- This duoidal structure induces a monoidal structure on $\mathbf{Chu}(R)$ based on $(\mathsf{Id}, \cdot)$.
- $R$-residual monad-comonad interaction laws are monoid objects of $\mathbf{Chu}(R)$ wrt this monoidal structure.

# General residual interaction laws

- Instead of an endofunctor category, one can consider *any* duoidal category $(\mathcal{D}, I, \diamond, J, \star)$.
- Given a monoid object $(R, \eta^R, \mu^R)$ wrt. $(I, \diamond)$, we get a $(I, \diamond)$-based monoidal structure on $\mathbf{Chu}(R)$.
- An $R$-residual monoid-comonoid interaction law is a monoid object of $\mathbf{Chu}(R)$.
- Explicitly, it is given by a monoid $(T, \eta, \mu)$, a comonoid $(D, \varepsilon, \delta)$ and a map $\psi : T \star D \to R$ such that